

On the Distribution of Long-Term Time Averages on Symbolic Space

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The pressure was studied in a rather abstract theory as an important notion of the thermodynamic formalism. The present paper gives a more concrete account in the case of symbolic spaces, including subshifts of finite type. We relate the pressure of an interaction function Φ to its long-term time averages through the Hausdorff and packing dimensions of the subsets on which Φ has prescribed long-term time-average values. Functions Φ with values in \mathbb{R}^d are considered. For those Φ depending only on finitely many symbols, we get complete results, unifying and completing many partial results.

KEY WORDS: Symbolic dynamics; Dimensions; Gibbs measures.

1. INTRODUCTION

Consider the Ising model whose energy of spin system is assumed to be

$$E_N(x) = -J \sum_{\langle j, k \rangle} x_j x_k - mH \sum_j x_j$$

where J is a positive constant, m the magnitude of the atomic magnetic moments and H the external magnetic field; the first sum is over all pairs of nearest neighbor spins. The model may be generalized by removing the

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restriction to nearest neighbor interactions to k -range interactions ($k \geq 2$). For the generalized model, the energy can be rewritten as

$$E_N(x) = \sum_{j=0}^{N-1} \Phi(T^j x) + \text{errors}$$

where T is the shift map on the configuration space and $\Phi(x) = \Phi(x_1, \dots, x_k)$ is a function which depends only on the first k coordinates of x . The ‘‘Ising problem’’ to calculate the free energy per spin F in the thermodynamic limit:

$$F = -kT \lim_{N \rightarrow \infty} \frac{1}{N} \log \sum_{x_1, \dots, x_N} e^{-(1/kT) E_N(x)}$$

and other derived quantities as the energy and the specific heat at constant field, the magnetization and susceptibility.

It is true that, in principle, an algebraic method provides a way to calculate F . However, it is not very effective when k is large because the maximal eigenvalue of a big matrix must be evaluated. In this paper, we present a way to relate F to the long-term time average of Φ defined by

$$\sigma_\Phi(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \Phi(T^j x)$$

or more exactly to the Hausdorff dimensions of the subsets $\{x : \sigma_\Phi(x) = \alpha\}$ of prescribed long-term time average values (see the formula at the end of this Introduction). We prove an exact formula for the dimension in many interesting cases (see Theorem 1 and Theorem 2).

The paper is written from a dynamical system point of view where we use the terminology pressure instead of the free energy. Then we raise our problem to study in the following way.

For a dynamical system $T: X \rightarrow X$ and a finite number of subsets A_1, \dots, A_d in X , how often does a point $x \in X$ go into A_j for each $1 \leq j \leq d$? This is a *multi* recurrence problem, which was rarely treated before. We shall tackle the problem by using a method different from the thermodynamical formalism, which was usually used in the case $d = 1$. Indeed, the thermodynamical formalism is a good theoretical method, but is not very practical because the pressure function is difficult to effectively calculate. For this reason we avoid to use it. It turns out that the results obtained in this paper will actually provide a new method to explicitly calculate the pressure functions for many finite range interactions on a one-dimensional system of lattice particles.

More generally, let $\Phi: X \rightarrow \mathbb{R}^d$ be a vector valued function. We would like to know the possible values of the following limits, if exist, for different points x

$$\sigma_\Phi(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x)$$

We call $\sigma_\Phi(x)$ the *long-term time average* or *recurrence* of x relative to Φ . We also would like to measure the size of the set of points x such that $\sigma_\Phi(x)$ is equal to a prescribed value. To be more precise, let

$$L_\Phi = \{ \alpha \in \mathbb{R}^d : \alpha = \sigma_\Phi(x) \text{ for some } x \in \mathbb{R}^d \}$$

For $\alpha \in L_\Phi$, let

$$E_\Phi(\alpha) = \{ x \in X : \sigma_\Phi(x) = \alpha \}$$

What is the limit set L_Φ ? How big is the set $E_\Phi(\alpha)$ for each $\alpha \in L_\Phi$?

The answers to these questions depend upon the dynamical system (X, T) . In the present paper we will discuss them in the special case of symbolic dynamical space (Σ, T) where $\Sigma = \{0, 1\}^{\mathbb{N}}$, T is the shift on Σ . It will be proved that for any continuous function Φ , L_Φ is a non-empty compact convex set and that the Hausdorff dimension and the packing dimension of $E_\Phi(\alpha)$ are equal to a certain concave function of α (Theorem 4). When Φ is locally constant, we will give a more precise description of the set L_Φ and prove a formula for the dimension of $E_\Phi(\alpha)$ in a variational form (Theorems 2 and 3). Closed form formulas are found in some special cases (Theorem 1, see also the examples at the end of the paper).

There is a way to interpret (Σ, T) as an interval mapping system by taking $X = [0, 1)$, $Tx = 2x \pmod{1}$. Because every real number $x \in [0, 1)$ is developed dyadically into $x = \sum_{n=1}^{\infty} x_n/2^n$ ($x_n = 0$ or 1). Without confusion, at least for dyadic irrational numbers, we will write $x = (x_n)$. Recall that Σ can be equipped with the metric $\rho(x, y) = 2^{-n(x, y)}$ where $n(x, y) = \inf \{ n \geq 1 : x_n \neq y_n \}$ and that the shift transformation T on Σ is defined by $(x_n)_{n \geq 1} \mapsto (x_{n+1})_{n \geq 1}$. The space Σ being a metric space, different notions of dimensions may be defined in the usual way (ref. 27, see also refs. 11 and 24). We shall use $\dim_H A$ and $\dim_P A$ to denote respectively the Hausdorff dimension and the Packing dimension of a set A .

The first historic example would be the following. For $\alpha \in [0, 1]$, let

$$E(\alpha) = \left\{ x = (x_n) \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j = \alpha \right\}$$

A. S. Besicovitch⁽¹⁾ and H. G. Eggleston⁽¹⁰⁾ considered these sets $E(\alpha)$ and obtained that

$$\dim_H E(\alpha) = h(\alpha) + h(1 - \alpha)$$

where $h(\alpha) = -\alpha \log_2 \alpha$ with $\log_2 \alpha = \log \alpha / \log 2$. It is noticed that this example corresponds to $\Phi(x) = 1_{[1/2, 1]}$, the characteristic function of the interval $[1/2, 1]$.

A natural generalization of the above case will be studied and a complete answer will be given, which is the first result in the paper. Let $k \geq 2$ be an integer and let $\alpha_1, \alpha_2, \dots, \alpha_k$ be k real numbers in $[0, 1]$ such that $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k$. Let

$$E(\alpha_1, \alpha_2, \dots, \alpha_k) = \left\{ x = (x_n) \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j \cdots x_{j+i-1} = \alpha_i, 1 \leq i \leq k \right\}$$

Let us also introduce the function

$$A(\alpha_1, \alpha_2, \dots, \alpha_k) = 2h(\alpha_{k-1} - \alpha_k) + h(\alpha_k) - h(\alpha_{k-1}) - h(1 - \alpha_1) + \sum_{j=0}^{k-2} h(\alpha_j - 2\alpha_{j+1} + \alpha_{j+2})$$

(where $\alpha_0 = 1$). Let $\{\alpha_j\}_{j=0}^k$ be a sequence of real numbers. We say it is *convex* if $\alpha_i - 2\alpha_{i+1} + \alpha_{i+2} \geq 0$ ($\forall 0 \leq i \leq k-2$),

Theorem 1. Let $E(\alpha_1, \alpha_2, \dots, \alpha_k)$ be defined as above, where $1 = \alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_k \geq 0$.

(1) If $\{\alpha_j\}_{j=0}^k$ is convex, then $E(\alpha_1, \alpha_2, \dots, \alpha_k) \neq \emptyset$ and

$$\dim_H E(\alpha_1, \alpha_2, \dots, \alpha_k) = \dim_P E(\alpha_1, \alpha_2, \dots, \alpha_k) = A(\alpha_1, \alpha_2, \dots, \alpha_k)$$

(2) If $\{\alpha_j\}_{j=1}^k$ is not convex, then $E(\alpha_1, \alpha_2, \dots, \alpha_k) = \emptyset$.

The above situation corresponds to the function:

$$\Phi(x) = (x_1, x_1 x_2, \dots, x_1 x_2 \cdots x_k), \quad (x = (x_n)_{n \geq 1})$$

For a general function Φ which depends only on a finite number of coordinates, we are also able to provide a satisfactory answer. Let Δ_k be the

compact convex set of all probability vectors $p = p(\cdot)$ defined on $\Sigma_k = \{0, 1\}^k$ satisfying the restriction

$$\begin{aligned} p(x_1, \dots, x_{k-1}, 0) + p(x_1, \dots, x_{k-1}, 1) \\ = p(0, x_1, \dots, x_{k-1}) + p(1, x_1, \dots, x_{k-1}) \end{aligned}$$

(If $k = 1$, there is no restriction). Define a map $\varphi: \Delta_k \rightarrow \mathbb{R}^d$ by

$$\varphi(p) = \sum_{x \in \Sigma_k} p(x) \Phi(x)$$

Theorem 2. Suppose that $\Phi: \Sigma \rightarrow \mathbb{R}^d$ is a function which depends only upon the first k coordinates ($k \geq 1$). Then

- (1) $L_\Phi = \varphi(\Delta_k)$.
- (2) For $\alpha \in L_\Phi$, we have

$$\dim_H E_\Phi(\alpha) = \dim_P E_\Phi(\alpha) = \max_{p: p \in \Delta_k, \varphi(p) = \alpha} H(p)$$

where

$$H(p) = \sum_{x_1, \dots, x_k} p(x_1, \dots, x_k) \log_2 \frac{p(x_1, \dots, x_{k-1}, 0) + p(x_1, \dots, x_{k-1}, 1)}{p(x_1, \dots, x_k)}$$

A. Bisbas *et al.* have studied the special case $\Phi(x) = x_1 x_2 \cdots x_k$ in a different way.^(3,4) We shall see that the result for this special case may be deduced from either Theorem 1 or Theorem 2. The case $\Phi(x) = ((1 - x_1)(1 - x_2), (1 - x_1)x_2, x_1(1 - x_2), x_1x_2)$ studied by P. Billingsley in ref. 2 is a direct consequence of Theorem 2. We shall see these in Section 8.

The formula in Theorem 2 is not explicit as that in Theorem 1. But for concrete cases, the maximum in the theorem can be computed as an explicit function of α and the domain L_Φ may also be explicitly described (see Section 8).

Our third result is a formal solution to the problem for those Φ which are Hölder continuous in the sense that $|\Phi(x) - \Phi(y)| \leq c\delta^{n(x,y)}$ for some constants $c > 0$, $0 < \delta < 1$. For $\beta \in \mathbb{R}^d$, let

$$P_\Phi(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[2^n \int_\Sigma \exp \left\langle \beta, \sum_{j=0}^{n-1} \Phi(T^j x) \right\rangle dx \right]$$

It is known that the above limit exists and the function $P_\Phi(\cdot)$ is analytic and convex (see [32]). We call $P_\Phi(\beta)$ the *pressure function* of Φ .

Theorem 3. Suppose that $\Phi: \Sigma \rightarrow \mathbb{R}^d$ is a Hölder function. If $\alpha = \nabla P_\Phi(\beta)$ for some $\beta \in \mathbb{R}^d$, then $\alpha \in L_\Phi$ and

$$\dim_H E_\Phi(\alpha) = \dim_P E_\Phi(\alpha) = -\frac{1}{\log 2} (\langle \beta, \alpha \rangle - P_\Phi(\beta))$$

The theorem says that the image of \mathbb{R}^d under the gradient ∇P_Φ is a subset of L_Φ . It is usually a proper subset because boundary points of L_Φ may not be images of the gradient. The following result describes the situation for a continuous function Φ (without further regularity like Hölder continuity).

Theorem 4. Suppose that $\Phi: \Sigma \rightarrow \mathbb{R}^d$ is a continuous function.

- (1) L_Φ is a non-empty compact convex set.
- (2) For any $\alpha \in L_\Phi$, we have

$$\dim_H E_\Phi(\alpha) = \dim_P E_\Phi(\alpha) = A_\Phi(\alpha)$$

where $A_\Phi(\alpha)$ is a concave function.

Let us give immediately a definition of the function $A_\Phi(\alpha)$. For $n \geq 1$ and $\varepsilon > 0$, let $f(\alpha, n, \varepsilon)$ be the number of n -cylinders (see its definition in the next section) which contain a point x such that

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) - \alpha \right| < \varepsilon$$

We define

$$A_\Phi(\alpha) = \lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, \varepsilon)}{\log 2^n}$$

The proof of Theorem 3 uses Gibbs measures. It is classical but our consideration of vector valued function Φ is new. Gibbs measures are also used in the proof of Theorem 2 but in a unusual and indirect way. One of the basic ideas in proving Theorems 1, 2, and 4 is to approximate $E(\alpha_1, \alpha_2, \dots, \alpha_k)$ or $E_\Phi(\alpha)$ by a sequence of homogeneous Moran sets.

What we state above is a kind of multifractal analysis. But it is a little different from the multifractal analysis of measures to which the term “multifractal” is often attached. Let us mention^(7–9, 12, 13, 15–17, 20, 21, 25, 26, 28–31, 33) (it is far from exhaustive). Another kind of multifractal analysis was

engaged in ref. 22 (see more references herein) where functions rather than measures are studied.

The main feature of the present study consists in the following aspects. The function Φ (Theorem 4) is only continuous and the classical thermodynamical formalism doesn't work no longer. Indeed, if Φ is Hölder continuous, the thermodynamical formalism provides a formula involving the pressure function (Theorem 3). But there is no effective way to compute the gradient of the pressure and the formula is not practical. However, our formula in Theorem 2 reduces the difficulty of calculation of pressure to a concave programming problem on a convex set of *finite* dimension and this programming problem is resolvable in many interesting cases. It should be pointed out that even in the one dimensional case where Φ is real valued, there were few discussions on the boundary of L_Φ and it is actually a subtle question. Fortunately, formulas in Theorems 1, 2 and 4 are valid either for interior points or boundary points. When Φ is vector valued, it is worthy to study the shape of L_Φ which is not always as one imagines (Theorem 1, see also ref. 23). Both results in Theorem 3 and Theorem 4 provide us the following formula for pressure functions

$$P_\Phi(\beta) = \inf_{\alpha} (\langle \alpha, \beta \rangle + \log 2 \cdot A(\alpha))$$

In the case where Φ depends only upon the first k coordinates (Φ is a finite range interaction), $A(\alpha)$ can be calculated by the formula in Theorem 2. Therefore, when the sets $\{p : \varphi(p) = \alpha\}$ is well understood, we may find an explicit formula for $A(\alpha)$ and then an explicit formula for the pressure function $P_\Phi(\beta)$.

The materials are organized as follows. In Section 2, we introduce some notation and present necessary known results which will be useful in the sequel and to which the reader is asked to refer to when necessary. In this introduction, the results are presented in an increasing order of generality. But their proofs will be presented in an inverse order. So, Theorems 4, 3, 2 and 1 will be respectively proved in Sections 3, 4, 5 and 6. The Section 7 is devoted to applications of Theorem 3 and the Section 8 to applications of Theorems 1 and 2. In the last Section 9, we point out how to generalize Theorems 2, 3 and 4 to subshifts of finite type.

2. NOTATION AND PRELIMINARY

We first give a list of notation which may be referred as to when it is necessary. Then, for the convenience of the reader, we mention three known results which will be useful.

The following notation will be used.

Σ_k	$\{0, 1\}^k$. Sequences in Σ_k are called words of length k
$x _n$	$(x_j)_{j=1}^n$ if $x = (x_j) \in \Sigma$. We sometimes write $x _n = x_1 \cdots x_n$
uv	$u_1 \cdots u_n v_1 \cdots v_m$ if $u = u_1 \cdots u_n \in \Sigma_n$ and $v = v_1 \cdots v_m \in \Sigma_m$
$I_n(x)$	n -cylinder consisting of y such that $y _n = x _n$.
	We also write $I(x _n)$
$\text{var}_n(\Phi)$	$\sup_{x _n = y _n} \Phi(x) - \Phi(y) $, $ \cdot $ denoting the Euclidean norm
$V_n(\Phi)$	$\sum_{j=1}^n \text{var}_n(\Phi)$
$S_n(\Phi, x)$	$\sum_{j=0}^{n-1} \Phi(T^j x)$
$A_n(\Phi, x)$	$\frac{1}{n} S_n(\Phi, x)$
$\sigma_\Phi(x)$	$\lim_{n \rightarrow \infty} A_n(\Phi, x)$
$P(\alpha, n, \varepsilon)$	$\{x \in \Sigma : A_n(\Phi, x) - \alpha < \varepsilon\}$
$F(\alpha, n, \varepsilon)$	$\{\omega \in \Sigma_n : I(\omega) \cap P(\alpha, n, \varepsilon) \neq \emptyset\}$
$f(\alpha, n, \varepsilon)$	$\text{Card } F(\alpha, n, \varepsilon)$

The following result concerns the existence of Gibbs measure.

Proposition 1. ^(6, 14, 32) Suppose that $\phi: \Sigma \rightarrow \mathbb{R}$ is a function of summable variation, i.e., $\sum_{n=1}^{\infty} \text{var}_n(\phi) < \infty$. There exists a unique probability T -invariant measure $\mu = \mu_\phi$ such that

$$c \leq \frac{\mu(I_n(x))}{\exp[-nP + \sum_{j=0}^{n-1} \phi(T^j x)]} < c^{-1} \quad (\forall x \in \Sigma, \forall n \geq 1)$$

where $c > 0$ and P are two constants.

The measure μ is called the *Gibbs measure* of ϕ . The constant P is also uniquely determined by ϕ and is called the *pressure* of ϕ .

For a general account of the different notions of dimensions, we can refer as to refs. 11, 24, 27 and 34. Recall that the Hausdorff dimension and

packing dimension are σ -stable and that if $\underline{\dim}_B$ and $\overline{\dim}_B$ denote respectively the lower and upper box dimension we have

$$\dim_H A \leq \underline{\dim}_B A, \quad \dim_P A \leq \overline{\dim}_B A \quad (\forall A)$$

Here is a very useful result for computation of dimensions, called Billingsley theorem.

Proposition 2 (ref. 27, p. 99, see also refs. 2 and 35). Let (X, d) be a (compact) metric space. Let μ be a Borel probability measure on X . For a Borel set $E \subset X$, we have $a \leq \dim_H E \leq b$ if

$$\mu(E) > 0, \quad E \subset \left\{ x \in X : a \leq \liminf_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r} \leq b \right\}$$

For a Borel set F , we have $c \leq \dim_P F \leq d$ if

$$\mu(F) > 0, \quad F \subset \left\{ x \in X : c \leq \limsup_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r} \leq d \right\}$$

($B_r(x)$ being the ball centered at x with radius r).

Now we discuss a class of Moran sets. Let $\{n_k\}_{k \geq 1}$ be a sequence of positive integers and $\{c_k\}_{k \geq 1}$ be a sequence of positive numbers satisfying $n_k \geq 2$, $0 < c_k < 1$, $n_1 c_1 \leq \delta$ and $n_k c_k \leq 1$ ($k \geq 2$), where δ is some positive number. Let

$$D = \bigcup_{k \geq 0} D_k \quad \text{with} \quad D_0 = \{\emptyset\}, \quad D_k = \{(i_1, \dots, i_k); 1 \leq i_j \leq n_j, 1 \leq j \leq k\}$$

If $\sigma = (\sigma_1, \dots, \sigma_k) \in D_k$, $\tau = (\tau_1, \dots, \tau_m) \in D_m$, we define $\sigma * \tau = (\sigma_1, \dots, \sigma_k, \tau_1, \dots, \tau_m)$.

Suppose J be a closed interval of length δ . A collection $\mathcal{F} = \{J_\sigma : \sigma \in D\}$ of closed subintervals of J is said to have a *homogeneous Moran structure* if it satisfies

- (1) $J_\emptyset = J$;
- (2) For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma * 1}, J_{\sigma * 2}, \dots, J_{\sigma * n_{k+1}}$ are subintervals of J_σ and $J_{\sigma * i} \cap J_{\sigma * j} = \emptyset$ ($i \neq j$) where $\overset{\circ}{A}$ denotes the interior of A ;
- (3) For any $k \geq 1$ and any $\sigma \in D_{k-1}$, $1 \leq j \leq n_k$, we have

$$\frac{|J_{\sigma * j}|}{|J_\sigma|} = c_k$$

where $|A|$ denotes the diameter of A .

Suppose that \mathcal{F} is a collection of closed subintervals of J having homogeneous Moran structure, $E(\mathcal{F}) := \bigcap_{k \geq 1} \bigcup_{\sigma \in D_k} J_\sigma$ is called a *homogeneous Moran set* determined by \mathcal{F} and the intervals in $\mathcal{F}_k = \{J_\sigma; \sigma \in D_k\}$ are called the k -order fundamental intervals of $E(\mathcal{F})$ and J is called the original interval of $E(\mathcal{F})$. It can be seen from above definition that for any fixed $J, \{n_k\}_{k \geq 1}, \{c_k\}_{k \geq 1}$, if the positions of k -order fundamental intervals are changed, we get different homogeneous Moran sets. We use $\mathcal{M}(J, \{n_k\}, \{c_k\})$ to denote the collection of all such homogeneous Moran sets determined by $J, \{n_k\}_{k \geq 1}, \{c_k\}_{k \geq 1}$. One may refer to ref. 18 and 19 for more informations about homogeneous Moran sets. For the purpose of the present paper, we only need a simplified version of a result contained in ref. 18, whose simpler proof will be given here for the convenience of the reader.

Proposition 3. For any $E \in \mathcal{M}(J, \{n_k\}, \{c_k\})$, we have

$$\dim_H E \geq \liminf_{n \rightarrow \infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}}$$

Proof. Denote by t the right hand side of the above inequality. Suppose $t > 0$. Let μ be the probability measure concentrated on E such that $\mu(A) = (n_1 n_2 \cdots n_k)^{-1}$ for any $A \in \mathcal{F}_k$. Let $0 < s < t$. By the definition of t , there exists $c > 0$ such that

$$n_1 n_2 \cdots n_k (c_1 c_2 \cdots c_{k+1} n_{k+1})^s \geq c \quad (\forall k \geq 1)$$

Let $U \subset [0, 1]$ be an arbitrary closed interval with $|U| \leq c_1$. There exists a positive integer k such that $c_1 c_2 \cdots c_{k+1} \leq |U| < c_1 c_2 \cdots c_k$. It follows that

- (i) U intersects at most $3|U|/c_1 c_2 \cdots c_{k+1}$ $(k+1)$ -order fundamental intervals;
- (ii) U intersects at most 2 k -order fundamental intervals.

By using the inequality $\min(a, b) \leq a^{1-s} b^s$ ($0 \leq s \leq 1$), we have

$$\begin{aligned} \mu(U) &\leq \min\left(\frac{2}{n_1 n_2 \cdots n_k}, \frac{3|U|}{c_1 c_2 \cdots c_{k+1} n_{k+1}} \times \frac{1}{n_1 n_2 \cdots n_k}\right) \\ &\leq \frac{1}{n_1 n_2 \cdots n_k} \left(\frac{3|U|}{c_1 c_2 \cdots c_{k+1} n_{k+1}}\right)^s 2^{1-s} \\ &\leq \frac{1}{c} 3^s 2^{1-s} |U|^s \leq \frac{6}{c} |U|^s \end{aligned}$$

This implies $\dim_H E \geq s$ then $\dim_H E \geq t$. ■

We can define homogenous Moran set in Σ by identifying cylinders with intervals. The same result holds. It is actually this result that we shall use.

3. PROOF OF THEOREM 4

We divide the proof into small steps. There is a simpler argument for proving that L_Φ is a non-empty convex set. But we are content with an elementary and direct proof.

Step 1. L_Φ is non-empty and bounded. For any p -periodic point x , i.e., $T^p x = x$, we have $\sigma_\Phi(x) = A_p(\Phi, x)$. So, L_Φ contains $A_p(\Phi, x)$. The boundedness of L_Φ is implied by the boundedness of Φ .

Step 2. Closedness of L_Φ . Suppose $\alpha_i \in L_\Phi$ such that $\lim_{i \rightarrow \infty} \alpha_i = \alpha$. We want to prove $\alpha \in L_\Phi$. We can find a sequence of points $x^{(i)} \in \Sigma$ and a sequence of integers $n_i \uparrow \infty$ such that $x^{(i)} \in P(\alpha_i, n_i, 2^{-i})$. Let $\omega_i = x^{(i)}|_{n_i}$. Choose $m_i = 2^{n_i+1}$. Then define a sequence in Σ as follows

$$\omega = \underbrace{\omega_1 \omega_1 \cdots \omega_1}_{m_1} \underbrace{\omega_2 \omega_2 \cdots \omega_2}_{m_2} \underbrace{\omega_3 \omega_3 \cdots \omega_3}_{m_3} \cdots$$

We are going to show that $\sigma_\Phi(\omega) = \alpha$. Following the construction of ω , we cut the set of non-negative integers into disjoint groups such that each of the first m_1 groups, noted $N_{1,1}, \dots, N_{1,m_1}$, has cardinal n_1 and each of the next m_2 groups, noted $N_{2,1}, \dots, N_{2,m_2}$, has cardinal n_2 and so on. For any $i \geq 1$ and any $1 \leq \ell \leq m_i$, we have

$$\left| \sum_{j \in N_{i,\ell}} \Phi(T^j \omega) - S_{n_i}(\Phi, x^{(i)}) \right| \leq V_{n_i}(\Phi)$$

For any n sufficiently large ($n > m_1 n_1$), there are unique integers k and $0 \leq q < m_{k+1}$ such that

$$m_1 n_1 + \cdots + m_k n_k + q n_{k+1} \leq n < m_1 n_1 + \cdots + m_k n_k + (q + 1) n_{k+1}$$

From the above obtained inequality, it follows that

$$\begin{aligned} & \left| \sum_{j=0}^{m_1 n_1 + \cdots + m_k n_k + q n_{k+1} - 1} \Phi(T^j \omega) - \sum_{i=1}^k m_i S_{n_i}(\Phi, x^{(i)}) - q S_{n_{k+1}}(\Phi, x^{(k+1)}) \right| \\ & \leq \sum_{i=1}^k m_i V_{n_i}(\Phi) + q V_{n_{k+1}}(\Phi) \end{aligned}$$

Recall the following elementary result, called Stokes theorem. Let $\{a_i\}$ be a sequence of real numbers and $\{b_i\}$ be a sequence of positive numbers such that $\sum_{i=1}^n b_i \rightarrow \infty$. Suppose $\lim_i (a_i/b_i) = \alpha$. Then $\lim_n (\sum_{i=1}^n a_i / \sum_{i=1}^n b_i) = \alpha$. From the last inequality, the fact $n^{-1}V_n(\Phi) \rightarrow 0$ and the Stokes theorem, it follows that the subsequence $A_n(\Phi, \omega)$ ($n = m_1n_1 + \dots + m_kn_k + qn_{k+1}$) tends to α . In order to pass the subsequence through to the whole sequence, it suffices to notice that

$$\sum_{j=m_1n_1 + \dots + m_kn_k + qn_{k+1}}^{n-1} \Phi(T^j\omega) = O(n_{k+1}) = O(\log m_k) = o(n)$$

and that

$$\lim_{n \rightarrow \infty} \frac{n}{m_1n_1 + \dots + m_kn_k + qn_{k+1}} = 1 \quad \blacksquare$$

Step 3. Convexity of L_Φ . It suffices to show the rational convexity in the sense that if $\alpha, \beta \in L_\Phi$ and p, q are positive integers, then $(p\alpha + q\beta)/(p + q) \in L_\Phi$. Take $x \in E_\Phi(\alpha)$ and $y \in E_\Phi(\beta)$. For $n \geq 1$, construct a finite sequence

$$\omega_n = \underbrace{x|_n \cdots x|_n}_p \underbrace{y|_n \cdots y|_n}_q$$

Then construct an infinite sequence $\omega = \omega_1\omega_2 \cdots$. As in Step 2, we can see that for any $n \geq 1$,

$$|S_{(p+q)n}(\Phi, \omega) - pS_n(\Phi, x) - qS_n(\Phi, y)| \leq (p + q) V_n(\Phi)$$

This, together with a similar but simpler argument as in Step 2, allows us to get $\sigma_\Phi(\omega) = (p\alpha + q\beta)/(p + q)$. \blacksquare

Step 4. For $\alpha \in L_\Phi$, we have

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, \varepsilon)}{\log 2^n} = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \varepsilon)}{\log 2^n} (= : A_\Phi(\alpha))$$

We want to show that as a sequence of n , $\log f(\alpha, n, \varepsilon)$ shares a kind of subadditivity. That means, for any $\varepsilon > 0$, there is a N such that

$$[f(\alpha, n, \varepsilon)]^m \leq f(\alpha, nm, 2\varepsilon) \quad (\forall n \geq N, \forall m \geq 1)$$

In fact, suppose $\omega_1, \dots, \omega_m \in F(\alpha, n, \varepsilon)$. Let $\omega = \omega_1 \cdots \omega_m$. If $x \in I(\omega)$, again by the argument in Step 2 we get

$$|S_{nm}(\Phi, x) - nm\alpha| \leq nm\varepsilon + mV_n(\Phi)$$

It follows that $\omega \in F(\alpha, nm, \varepsilon + n^{-1}V_n)$. Consequently, since different choices $(\omega_1, \dots, \omega_m)$ produce different ω ,

$$[f(\alpha, n, \varepsilon)]^m \leq f(\alpha, nm, \varepsilon + n^{-1}V_n(\Phi))$$

Take a sufficiently large N so that $n^{-1}V_n(\Phi) \leq \varepsilon$ ($n \geq N$). Thus the claimed subadditivity is proved. By using the subadditivity, it is easy to see that

$$\limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \varepsilon)}{\log 2^n} \leq \liminf_{n \rightarrow \infty} \frac{\log f(\alpha, n, 2\varepsilon)}{\log 2^n}$$

which finishes Step 4. ■

Step 5. For $\alpha \in L_\Phi$, we have $\dim_P E_\Phi(\alpha) \leq A_\Phi(\alpha)$. Let

$$G(\alpha, m, \varepsilon) = \bigcap_{n=m}^{\infty} \{x \in \Sigma : |A_n(\Phi, x) - \alpha| < \varepsilon\}$$

It is clear that for any $\varepsilon > 0$,

$$E_\Phi(\alpha) \subset \bigcup_{m=1}^{\infty} G(\alpha, m, \varepsilon)$$

Note that if $n \geq m$, $G(\alpha, m, \varepsilon)$ is covered by the union of all cylinders $I(\omega)$ with $\omega \in F(\alpha, n, \varepsilon)$ whose total number is $f(\alpha, n, \varepsilon)$. Therefore we have the following estimate

$$\overline{\dim}_B G(\alpha, m, \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \varepsilon)}{\log 2^n} \quad (\forall \varepsilon > 0, \forall m \geq 1)$$

On the other hand, by using the σ -stability of the packing dimension, we have

$$\begin{aligned} \dim_P E_\Phi(\alpha) &\leq \dim_P \left(\bigcup_{m=1}^{\infty} G(\alpha, m, \varepsilon) \right) \leq \sup_m \dim_P G(\alpha, m, \varepsilon) \\ &\leq \sup_m \overline{\dim}_B G(\alpha, m, \varepsilon) \end{aligned}$$

This, together with the above estimate and Step 4, leads to the desired result. ■

Step 6. For $\alpha \in L_\Phi$, we have $\dim_H E_\Phi(\alpha) \geq A_\Phi(\alpha)$. Given $\delta > 0$. By Step 4, there are $\ell_j \uparrow \infty$ and $\varepsilon_j \downarrow 0$ such that

$$f(\alpha, \ell_j, \varepsilon_j) > 2^{\ell_j(A_\Phi(\alpha) - \delta/2)}$$

Write simply $F_{\ell_j} = F(\alpha, \ell_j, \varepsilon_j)$ and $f_{\ell_j} = f(\alpha, \ell_j, \varepsilon_j)$. Define a new sequence $\{\ell_j^*\}$ in the following manner

$$\underbrace{\ell_1, \dots, \ell_1}_{N_1}; \underbrace{\ell_2, \dots, \ell_2}_{N_2}; \dots; \underbrace{\ell_j, \dots, \ell_j}_{N_j}; \dots$$

where N_j is defined recursively by

$$N_j = 2^{\ell_{j+1} + N_{j-1}} \quad (j \geq 2); \quad N_1 = 1$$

Let $n_j = f_{\ell_j^*}$ and $c_j = 2^{-\ell_j^*}$. Define

$$\Theta = \prod_{j=1}^{\infty} F_{\ell_j^*}$$

We are going to show that $\Theta \subset E_\Phi(\alpha)$. In fact, for any $n (> \ell_1)$, there is a unique integer $J(n)$ such that

$$\sum_{i=1}^{J(n)} \ell_i^* \leq n < \sum_{i=1}^{J(n)+1} \ell_i^*$$

The choice of N_j implies that $\ell_{k+1} = o(N_k)$. It follows that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{J(n)} \ell_i^*}{\sum_{i=1}^{J(n)+1} \ell_i^*} = 1$$

Let $x = (x_n) \in \Theta$. Let A_j be the set of integers between $\ell_1^* + \dots + \ell_{j-1}^* + 1$ and $\ell_1^* + \dots + \ell_j^*$. We have

$$\begin{aligned} S_n(\Phi, x) &= \sum_{j=1}^{J(n)} \sum_{k \in A_j} \Phi(T^k x) + O(\ell_{J(n)+1}^*) \\ &= \sum_{j=1}^{J(n)} \ell_j^*(\alpha + o(1)) + o(n) \\ &= \alpha n + o(n) \end{aligned}$$

It follows that $x \in E_\alpha$. Observe that Θ is a homogeneous Moran set in Σ . More precisely $\Theta \in \mathcal{M}(\Sigma, \{n_j\}, \{c_j\})$. By Proposition 3, we have

$$\begin{aligned} \dim_H \Theta &\geq \liminf_{k \rightarrow \infty} \frac{\log(n_1 \cdots n_k)}{-\log(c_1 \cdots c_k c_{k+1} n_{k+1})} \\ &\geq \liminf_{k \rightarrow \infty} \frac{\log(f_{\ell_1^*} \cdots f_{\ell_k^*})}{\log(2^{\ell_1^* + \ell_k^* + \ell_{k+1}^*})} \\ &= \liminf_{k \rightarrow \infty} \frac{\log(f_{\ell_1^*} \cdots f_{\ell_k^*})}{\log(2^{\ell_1^* + \ell_k^*})} \\ &\geq A_\Phi(\alpha) - \delta \quad \blacksquare \end{aligned}$$

Step 7. $A_\Phi: L_\Phi \rightarrow [0, 1]$ is concave. For $\alpha \in L_\Phi$, it is evident that $A_\Phi(\alpha) \geq 0$. Since $f(\alpha, n, \varepsilon) \leq 2^n$, $A_\Phi(\alpha) \leq 1$. Now let $\alpha, \beta \in L_\Phi$. Let p, q be two positive integers. By the subadditivity proved in Step 4, for large n we have

$$[f(\alpha, n, \varepsilon)]^p [f(\beta, n, \varepsilon)]^q \leq f(\alpha, np, 2\varepsilon) f(\beta, nq, 2\varepsilon)$$

Let $u \in F(\alpha, np, 2\varepsilon)$ and $v \in F(\beta, nq, 2\varepsilon)$. Take a point $x \in I(uv)$. We have

$$|S_{(p+q)n}(\Phi, x) - np\alpha - nq\beta| \leq 2\varepsilon n(p+q) + V_{np}(\Phi) + V_{nq}(\Phi)$$

It follows that if n is sufficient large, $uv \in F((p\alpha + q\beta)/(p+q), n(p+q), 3\varepsilon)$. Consequently, for large n we have

$$f(\alpha, np, 2\varepsilon) f(\beta, nq, 2\varepsilon) \leq f\left(\frac{p\alpha + q\beta}{p+q}, n(p+q), 3\varepsilon\right)$$

By the result in Step 4, we can get

$$\frac{p}{p+q} A_\Phi(\alpha) + \frac{q}{p+q} A_\Phi(\beta) \leq A_\Phi\left(\frac{p}{p+q} \alpha + \frac{q}{p+q} \beta\right)$$

We could say that we have proved the rational concavity of the (bounded) function A_Φ . However, the concavity of A_Φ is a consequence of its rational concavity. To see this, it suffices to consider the one-dimensional case. \blacksquare

Let us make some remarks on Theorem 4.

Remark 1. L_Φ is the closure of the set of all averages $A_p(\Phi, x)$ for different p -periodic points x ($p \geq 1$).

We have seen in Step 2 that all such averages are in L_Φ . So, we have to check that any point $\alpha \in L_\Phi$ may be approximated by such averages. Suppose $\alpha = \sigma_\Phi(x)$ for some x . For $m \geq 1$, let $x^{(m)}$ be the m -periodic point such that $x^{(m)}|_m = x|_m$. Then $\alpha = \lim_{m \rightarrow \infty} A_m(\Phi, x^{(m)})$.

Remark 2. For $\alpha \in L_\Phi$, we have

$$\dim_H E_\Phi(\alpha) = \dim_H \bar{E}_\Phi(\alpha) = \dim_H \underline{E}_\Phi(\alpha)$$

where

$$\bar{E}_\Phi(\alpha) = \{x \in \Sigma : \limsup_n A_n(\Phi, x) = \alpha\}$$

$$\underline{E}_\Phi(\alpha) = \{x \in \Sigma : \liminf_n A_n(\Phi, x) = \alpha\}$$

Let $\tilde{E}(\alpha)$ be the set of all x such that α is a cluster point of $A_n(\Phi, x)$. Since Theorem 4 is now available, we have only to show that $\dim_H \tilde{E}(\alpha) \leq A_\Phi(\alpha) + \delta$ for any $\delta > 0$. By the result in Step 4, for some $\varepsilon > 0$ we have

$$\limsup_{n \rightarrow \infty} \frac{\log f(\alpha, n, \varepsilon)}{\log 2^n} \leq A_\Phi(\alpha) + \frac{\delta}{2}$$

Note that, $\tilde{E}(\alpha) \subset \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} G_n$ where G_n is the union of all $I(\omega)$ with $\omega \in F(\alpha, n, \varepsilon)$. Thus letting $s = A_\Phi(\alpha) + \delta$, for any $m \geq 1$ we have

$$\mathcal{H}_{2^{-m}}^s(\tilde{E}(\alpha)) \leq \sum_{n=m}^{\infty} f(\alpha, n, \varepsilon) 2^{-ns} \leq \sum_{n=1}^{\infty} f(\alpha, n, \varepsilon) 2^{-ns} \leq C \sum_{n=1}^{\infty} 2^{-(\delta/2)n} < \infty$$

4. PROOF OF THEOREM 3

The result of Theorem 3 was well-known for the one-dimensional case, i.e., $d=1$ (see ref. 13). The following proof for the high-dimensional case consists in introducing the family of energy functions $\phi_\beta = \langle \beta, \Phi \rangle$ ($\beta \in \mathbb{R}^d$) and considering the corresponding family of Gibbs measures $\mu_\beta := \mu_{\phi_\beta}$. Denote

$$D(\mu_\beta, x) = \lim_{n \rightarrow \infty} \frac{\log \mu_\beta(I_n(x))}{\log |I_n(x)|}$$

when the limit exists. By the Gibbs property of μ_β (Proposition 1), we get the following relation

$$D(\mu_\beta, x) = -\frac{1}{\log 2} \left(\lim_{n \rightarrow \infty} \frac{1}{n} \left\langle \beta, \sum_{j=0}^{n-1} \Phi(T^j x) \right\rangle - P_\Phi(\beta) \right)$$

Since the measure μ_β is ergodic, by Birkhoff ergodic theorem, μ_β -almost everywhere we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) = \int_\Sigma \Phi d\mu_\beta = \nabla P_\Phi(\beta)$$

(see refs. 5 and 32 for the second equality). Since $\alpha = \nabla P_\Phi(\beta)$, it follows that $\mu_\beta(E_\Phi(\alpha)) = 1$ and for any $x \in E_\Phi(\alpha)$,

$$D(\mu_\beta, x) = -\frac{1}{\log 2} (\langle \beta, \nabla P_\Phi(\beta) \rangle - P_\Phi(\beta))$$

Thus Theorem 3 follows from the Billingsley theorem (Proposition 2). ■

5. PROOF OF THEOREM 2

5.1. Lemmas

Let Δ_k^+ be the set of strictly positive probability vectors in Δ_k , i.e., $p \in \Delta_k$ such that $p(x) > 0$ ($\forall x \in \Sigma_k$).

Lemma 1. Δ_k^+ is a dense subset of Δ_k .

Proof. Since Δ_k is convex and $p_0 = (1/2^k, \dots, 1/2^k) \in \Delta$, for any $0 < \varepsilon < 1$ we have

$$(1 - \varepsilon) \Delta_k + \varepsilon p_0 \subset \Delta_k^+$$

We get the desired result by letting $\varepsilon \rightarrow 0$. ■

For any $\omega = (x_j)_{1 \leq j \leq n+k-1}$, we define

$$N_\omega(\varepsilon_1, \dots, \varepsilon_k) = \text{Card} \{ 1 \leq j \leq n : x_{j+\ell-1} = \varepsilon_\ell, 1 \leq \ell \leq k \}$$

Note that $N_\omega(\varepsilon_1, \dots, \varepsilon_k)$ is just the number of repetitions of the word $\varepsilon_1 \cdots \varepsilon_k$ in the word ω . The following observation is immediate but fundamental and is our starting point.

Lemma 2. For $x \in \Sigma$ and $n \geq 1$, we have

$$\sum_{j=0}^{n-1} \Phi(T^j x) = \sum_{\varepsilon_1, \dots, \varepsilon_k} N_{\omega_n}(\varepsilon_1, \dots, \varepsilon_k) \Phi(\varepsilon_1, \dots, \varepsilon_k)$$

where $\omega_n = (x_j)_{1 \leq j \leq n+k-1}$.

It follows that our problem is reduced to the study of the limits of $n^{-1}N_{\omega_n}(\varepsilon_1, \dots, \varepsilon_k)$ as $n \rightarrow \infty$. The possible limits will be described through the following lemma.

Lemma 3. For any $\delta > 0$, there is a constant $N > 0$ such that for every $\omega = (x_j)_{j=1}^{n+k-1} \in \Sigma_{n+k-1}$ with $n \geq N$, there exists a probability vector $p \in \Delta_k$ having the property that for all $(\varepsilon_1, \dots, \varepsilon_k) \in \Sigma_k$,

$$\left| \frac{N_{\omega}(\varepsilon_1, \dots, \varepsilon_k)}{n} - p(\varepsilon_1, \dots, \varepsilon_k) \right| < \delta, \quad p(\varepsilon_1, \dots, \varepsilon_k) \geq \frac{\delta}{2^{k+1}}$$

Proof. Denote $\omega' = (x_j)_{1 \leq j \leq (n-1)+k-1}$ and $\omega'' = (x_{j+1})_{1 \leq j \leq (n-1)+k-1}$. Observe the following relations

$$\begin{aligned} \sum_{\varepsilon_1, \dots, \varepsilon_k} N_{\omega}(\varepsilon_1, \dots, \varepsilon_k) &= n \\ N_{\omega}(\varepsilon_1, \dots, \varepsilon_{k-1}, 0) + N_{\omega}(\varepsilon_1, \dots, \varepsilon_{k-1}, 1) &= N_{\omega'}(\varepsilon_1, \dots, \varepsilon_{k-1}) \\ N_{\omega}(0, \varepsilon_1, \dots, \varepsilon_{k-1}) + N_{\omega}(1, \varepsilon_1, \dots, \varepsilon_{k-1}) &= N_{\omega''}(\varepsilon_1, \dots, \varepsilon_{k-1}) \end{aligned}$$

The first relation implies that $n^{-1}N_{\omega}(\cdot)$ is a probability vector and the last two relations imply

$$\begin{aligned} &|N_{\omega}(\varepsilon_1, \dots, \varepsilon_{k-1}, 0) + N_{\omega}(\varepsilon_1, \dots, \varepsilon_{k-1}, 1) \\ &\quad - N_{\omega}(0, \varepsilon_1, \dots, \varepsilon_{k-1}) - N_{\omega}(1, \varepsilon_1, \dots, \varepsilon_{k-1})| \leq 1 \end{aligned}$$

Now we deduce by contradiction that the first claimed inequality in the lemma holds. If the inequality didn't hold, there would be a positive number δ_0 , a sequence of integers $n_j \uparrow \infty$ and a sequence of words $\omega_{n_j} \in \Sigma_{n_j+k-1}$ such that for any $p \in \Delta_k$ we have

$$\left| \frac{N_{\omega_{n_j}}(\varepsilon_1, \dots, \varepsilon_k)}{n_j} - p(\varepsilon_1, \dots, \varepsilon_k) \right| \geq \delta_0 \quad (\text{some } (\varepsilon_1, \dots, \varepsilon_k))$$

Since the set of probability vectors is compact, we can suppose that $n_j^{-1}N_{\omega_{n_j}}(\cdot)$ converges. Denote its limit by \tilde{p} , which is in Δ_k by the above observation. Now note that the last inequality is violated by \tilde{p} , which is a contradiction. Thus we have proved that there exists $p \in \Delta_k$ such that

$$\left| \frac{N_{\omega_{n_j}}(\varepsilon_1, \dots, \varepsilon_k)}{n_j} - p(\varepsilon_1, \dots, \varepsilon_k) \right| < \frac{\delta}{2}$$

(For convenience, we take $\delta/2$ in place of δ). Let $p_0 = (1/2^k, \dots, 1/2^k)$ and $p' = (1 - \delta/2)p + (\delta/2)p_0$. It is easy to check that this p' is what we want. ■

The following lemma is actually already obtained in the proof of the preceding lemma.

Lemma 4. If $n_j^{-1}N_{\omega_{n_j}}(\cdot) \rightarrow p(\cdot)$, then $p \in \Delta_k$.

5.2. Proof of Theorem 2

Now we prove Theorem 2 by four steps.

Step 1. $L_\Phi \subset \varphi(\Delta_k)$. Suppose $\alpha \in L_\Phi$. Then there is a $x = (x_i) \in \Sigma$ such that $\sigma_\Phi(x) = \alpha$. We want to show $\alpha \in \varphi(\Delta_k)$. By Lemma 2, we have

$$\frac{1}{n} \sum_{j=0}^{n-1} \Phi(T^j x) = \sum_{\varepsilon_1, \dots, \varepsilon_k} n^{-1} N_{\omega_n}(\varepsilon_1, \dots, \varepsilon_k) \Phi(\varepsilon_1, \dots, \varepsilon_k)$$

where $\omega_n = (x_j)_{1 \leq j \leq n+k-1}$. There is certainly a convergent subsequence of $n^{-1}N_{\omega_n}(\cdot)$, whose limit p is in Δ_k by Lemma 4. It follows that $\alpha = \varphi(p)$. ■

Step 2. $\varphi(\Delta_k^+) \subset L_\Phi$. Actually, we will prove that if $\alpha = \varphi(p)$ with $p \in \Delta_k^+$, then

$$E_\Phi(\alpha) \neq \emptyset, \quad \dim_H E_\Phi(\alpha) \geq H(p)$$

For $\varepsilon_1, \dots, \varepsilon_k \in \Sigma_k$, write

$$t(\varepsilon_1, \dots, \varepsilon_k) = \frac{p(\varepsilon_1, \dots, \varepsilon_k)}{p(\varepsilon_1, \dots, \varepsilon_{k-1}, 0) + p(\varepsilon_1, \dots, \varepsilon_{k-1}, 1)}$$

Since p is strictly positive, t is a well defined function on Σ_k . Then, for any $n \geq 1$, define a function q_n on Σ_n as follows. If $n < k$, let

$$q_n(a_1, \dots, a_n) = \sum_{\varepsilon_1, \dots, \varepsilon_{k-n}} p(a_1, \dots, a_n, \varepsilon_1, \dots, \varepsilon_{k-n})$$

If $n \geq k$, let

$$q_n(a_1, \dots, a_n) = p(a_1, \dots, a_k) t(a_2, \dots, a_{k+1}) \cdots t(a_{n-k+1}, \dots, a_n)$$

Using the fact $t(\varepsilon_1, \dots, \varepsilon_{k-1}, 0) + t(\varepsilon_1, \dots, \varepsilon_{k-1}, 1) = 1$, it is easy to check that

$$q_1(0) + q_1(1) = 1$$

$$\sum_{\varepsilon_{n+1}} q_{n+1}(a_1, \dots, a_n, \varepsilon_{n+1}) = q_n(a_1, \dots, a_n)$$

$$\sum_{\varepsilon_1} q_{n+1}(\varepsilon_1, a_1, \dots, a_n) = q_n(a_1, \dots, a_n)$$

By the first two equalities above and the Kolmogorov consistent theorem, there exists a unique probability measure ν_p such that

$$\nu_p(I_n(a_1, \dots, a_n)) = q_n(a_1, \dots, a_n)$$

By the third equality, ν_p is T -invariant. It is clear, from the definition of q_n , that ν_p shares the Gibbs property relative to the energy function defined by

$$\psi(x) = \log t(x_1, \dots, x_k), \quad x = (x_j) \in \Sigma$$

The measure ν_p being ergodic (see ref. 6), according to the Birkhoff ergodic theorem, for ν_p -almost all $x \in \Sigma$ we have

$$\begin{aligned} \sigma_{\Phi}(x) &= \int_{\Sigma} \Phi \, d\nu_p \\ &= \sum_{\varepsilon_1, \dots, \varepsilon_k} \Phi(\varepsilon_1, \dots, \varepsilon_k) q_k(\varepsilon_1, \dots, \varepsilon_k) \\ &= \sum_{\varepsilon_1, \dots, \varepsilon_k} \Phi(\varepsilon_1, \dots, \varepsilon_k) p(\varepsilon_1, \dots, \varepsilon_k) \\ &= \varphi(p) = \alpha \end{aligned}$$

This implies $\nu_p(E_\phi(\alpha)) = 1$ and then $E_\phi(\alpha) \neq \emptyset$. The ergodic theorem also implies that for ν_p -almost all $x \in \Sigma$ we have

$$\lim_{n \rightarrow \infty} \frac{\nu_p(I_n(x))}{\log |I_n(x)|} = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} \frac{\psi(T^j x)}{-n \log 2} = -\frac{1}{\log 2} \int_{\Sigma} \psi \, d\nu_p = H(p)$$

So, the estimate on the dimension follows from the Billingsley theorem (Proposition 2). ■

Step 3. $\varphi(\Delta_k) \setminus \varphi(\Delta_k^+) \subset L_\phi$. We want to prove the same results as in the Step 2 for a non-negative vector p .

Let $\alpha = \varphi(p)$. By lemma 1, there is a sequence of $p_m \in \Delta_k^+$ such that $\lim_m p_m = p$. Obviously $\lim_m \alpha_m = \alpha$ where $\alpha_m = \varphi(p_m)$. By what we proved in Step 2,

$$\dim_H E_\phi(\alpha_m) \geq H(p_m), \quad (\forall m \geq 1)$$

Using this fact, we are going to construct a homogeneous Moran set contained in $E_\phi(\alpha)$. We claim first that for any $m \geq 1$, $\varepsilon > 0$ and $\delta > 0$, there exists an integer $n_{m, \varepsilon, \delta} \geq 1$ such that

$$f(\alpha_m, n, \varepsilon) \geq 2^{n(H(p_m) - \delta)} \quad (\forall n \geq n_{m, \varepsilon, \delta})$$

In fact, we have

$$E_\phi(\alpha_m) \subset \bigcup_{\ell=1}^{\infty} G(\alpha_m, \ell, \varepsilon)$$

where

$$G(\alpha_m, \ell, \varepsilon) = \bigcap_{n \geq \ell} P(\alpha_m, n, \varepsilon)$$

By the σ -stability of the Hausdorff dimension, there must be an $\ell = \ell(\alpha_m, \varepsilon, \delta)$ such that

$$\dim_H G(\alpha_m, \ell, \varepsilon) \geq \dim_H E_\phi(\alpha_m) - \frac{\delta}{2} \geq H(p_m) - \frac{\delta}{2}$$

Since for any $n \geq \ell$, $\{I(\omega)\}$ ($\omega \in F(m, n, \varepsilon)$) is a net cover of $G(\alpha_m, \ell, \varepsilon)$, we have

$$\liminf_{n \rightarrow \infty} \frac{\log f(m, n, \varepsilon)}{\log 2^n} \geq \underline{\dim}_B G(\alpha_m, \ell, \varepsilon) \geq H(p_m) - \frac{\delta}{2}$$

from which follows the claim.

Take now an increasing sequence $\ell_m \geq \ell(\alpha_m, 1/m, \delta)$ such that

$$f(\alpha_m, \ell_m, 1/m) \geq 2^{n(H(p_m) - \delta)}$$

Write simply $F_m = F(\alpha_m, \ell_m, 1/m)$ and $f_m = f(\alpha_m, \ell_m, 1/m)$. As in Step 6 of the proof of Theorem 4, define

$$\Theta = \prod_{j=1}^{\infty} F_{\ell_j^*}$$

We are going to show that $\Theta \subset E_{\Phi}(\alpha)$, which proves $E_{\Phi}(\alpha) \neq \emptyset$, and

$$\dim_H \Theta \geq \lim_{m \rightarrow \infty} H(p_m) - \frac{\delta}{2} \geq H(p) - \frac{\delta}{2}$$

In fact, for any $n (> \ell_1)$, let $J(n)$ be the unique integer such that

$$\sum_{i=1}^{J(n)} \ell_i^* \leq n < \sum_{i=1}^{J(n)+1} \ell_i^*$$

The choice of N_j implies that

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{J(n)} \ell_i^*}{\sum_{i=1}^{J(n)+1} \ell_i^*} = 1$$

Let $x = (x_n) \in \Theta$. Let A_j be the set of integers between $\ell_1^* + \dots + \ell_{j-1}^* + 1$ and $\ell_1^* + \dots + \ell_j^*$. We have

$$\begin{aligned} S_n(x) &= \sum_{j=1}^{J(n)} \sum_{k \in A_j} \Phi(T^k x) + o(n) \\ &= \sum_{j=1}^{J(n)} \ell_j^* (\alpha_j^* + o(1)) + o(n) \end{aligned}$$

where α_j^* is the sequence defined by α_j as ℓ_j^* is defined by ℓ_j . It follows that $x \in E_{\alpha}$. Observe that Θ is a homogeneous Moran set, i.e., $\Theta \in \mathcal{M}([0, 1], \{n_j\}, \{c_j\})$. The estimate $\dim_H \Theta \geq H(p) - \delta$ may be obtained in the same way as in the proof of Theorem 4. ■

By now, we have proved the first assertion of Theorem 2 and that for $\alpha \in L_{\Phi}$ we have

$$\dim_H E_{\Phi}(\alpha) \geq \max_{p \in A_k, \varphi(\alpha) = p} H(p)$$

Step 4. For $\alpha \in L_\Phi$, we have

$$\dim_P E_\Phi(\alpha) \leq \max_{p \in A_k, \Phi(\alpha) = p} H(p)$$

Denote the right hand side by $d(\alpha)$. For $\varepsilon > 0$ and $n \geq 1$, let

$$B(n, \varepsilon) = \bigcap_{m=n}^{\infty} \{x \in \Sigma: |m^{-1}S_m(\Phi, x) - \alpha| < \varepsilon\}$$

We have obviously

$$E_\Phi(\alpha) \subset \limsup_{n \rightarrow \infty} B(n, \varepsilon) \quad (\forall \varepsilon > 0)$$

By the σ -stability of the packing dimension and the inequality $\dim_P \leq \overline{\dim}_B$, we have only to show that

$$\liminf_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \overline{\dim}_B B(n, \varepsilon) \leq d(\alpha)$$

We are then led to find a cover of $B(n, \varepsilon)$ and estimate the number of cylinders in it. Take the cover consisting of all cylinders $I(x_1, \dots, x_{n+k-1})$ containing some $y \in B(n, \varepsilon)$. Estimate now $T(n, \varepsilon)$, the number of all these cylinders. In order to estimate $T(n, \varepsilon)$, we are going to divide the cover into several classes of cylinders and then to estimate the number of cylinders in each class. Let $\{n(\varepsilon_1, \dots, \varepsilon_k)\}$ be a system of 2^k non-negative integers such that

$$n(\varepsilon_1, \dots, \varepsilon_k) = N_\omega(\varepsilon_1, \dots, \varepsilon_k)$$

$$\left| \frac{1}{n} \sum_{\varepsilon_1, \dots, \varepsilon_k} n(\varepsilon_1, \dots, \varepsilon_k) \Phi(\varepsilon_1, \dots, \varepsilon_k) - \alpha \right| < \varepsilon$$

for some $\omega \in \Sigma_{n+k-1}$. We denote by $\mathcal{T}(\{n(\varepsilon_1, \dots, \varepsilon_k)\})$ the collection of all cylinders $I(\omega)$ with ω having the above mentioned property, and $\Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\})$ its cardinal. Let \mathcal{P}_k be the set of all possible systems $\{n(\varepsilon_1, \dots, \varepsilon_k)\}$. It is clear that the cardinal of \mathcal{P}_k is at most n^{2^k} . Then

$$T(n, \varepsilon) = \sum \Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\}) \leq n^{2^k} \max \Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\})$$

where the sum and the supremum are taken over the set \mathcal{P}_k . Thus

$$\frac{\log T(n, \varepsilon)}{\log 2^n} \leq \max \frac{\Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\})}{\log 2^n} + O\left(\frac{\log n}{n}\right)$$

Now let us estimate $\Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\})$ through some conveniently constructed invariant measure. Since $n(\varepsilon_1, \dots, \varepsilon_k) = N_\omega(\varepsilon_1, \dots, \varepsilon_k)$, by Lemma 3, we can find $p \in \Delta_k^+$ such that

$$\left| \frac{n(\varepsilon_1, \dots, \varepsilon_k)}{n} - p(\varepsilon_1, \dots, \varepsilon_k) \right| < \varepsilon, \quad p(\varepsilon_1, \dots, \varepsilon_k) > \frac{\varepsilon}{2^{k+1}}$$

As in Step 2, we construct a measure ν_p by using this p . For any $\omega = (x_i)_{i=1}^{n-k+1} \in \mathcal{F}(\{n(\varepsilon_1, \dots, \varepsilon_k)\})$, $N_\omega(\varepsilon_1, \dots, \varepsilon_k) = n(\varepsilon_1, \dots, \varepsilon_k)$. So, we have

$$\begin{aligned} \nu_p(I(\omega)) &= \frac{p(x_1, \dots, x_k)}{t(x_1, \dots, x_k)} \prod_{\varepsilon_1, \dots, \varepsilon_k} t(\varepsilon_1, \dots, \varepsilon_k)^{n(\varepsilon_1, \dots, \varepsilon_k)} \\ &\geq \frac{\varepsilon}{2^{k+1}} \prod_{\varepsilon_1, \dots, \varepsilon_k} t(\varepsilon_1, \dots, \varepsilon_k)^{n(\varepsilon_1, \dots, \varepsilon_k)} \end{aligned}$$

where $(x_i)_{i=1}^k = \omega|_k$. Let a denote the right hand side of the above inequality. Then

$$a \cdot \Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\}) \leq \nu_p \left(\bigcup_{\omega} I_{n+k-1}(\omega) \right) \leq 1$$

Combining the last two expressions gives

$$\Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\}) \leq \frac{1}{a} \leq \frac{2^{k+1}}{\varepsilon} \prod_{\varepsilon_1, \dots, \varepsilon_k} t(\varepsilon_1, \dots, \varepsilon_k)^{-n(\varepsilon_1, \dots, \varepsilon_k)}$$

Then

$$\begin{aligned} &\frac{\log \Gamma(\{n(\varepsilon_1, \dots, \varepsilon_k)\})}{\log 2^n} \\ &\leq O\left(\frac{\log \varepsilon}{n}\right) - \sum_{\varepsilon_1, \dots, \varepsilon_k} \frac{n(\varepsilon_1, \dots, \varepsilon_k)}{n} \log_2 t(\varepsilon_1, \dots, \varepsilon_k) \\ &\leq O\left(\frac{\log \varepsilon}{n}\right) - \sum_{\varepsilon_1, \dots, \varepsilon_k} p(\varepsilon_1, \dots, \varepsilon_k) \log_2 t(\varepsilon_1, \dots, \varepsilon_k) + O(-\varepsilon \log \varepsilon) \end{aligned}$$

Note that $\varphi(p)$ is near α in the sense that

$$\begin{aligned} |\varphi(p) - \alpha| &= \left| \sum_{\varepsilon_1, \dots, \varepsilon_k} p(\varepsilon_1, \dots, \varepsilon_k) \Phi(\varepsilon_1, \dots, \varepsilon_k) - \alpha \right| \\ &\leq \left| \sum_{\varepsilon_1, \dots, \varepsilon_k} \frac{n(\varepsilon_1, \dots, \varepsilon_k)}{n} \Phi(\varepsilon_1, \dots, \varepsilon_k) - \alpha \right| + 2^k \varepsilon \|\Phi\| < (2^k \|\Phi\| + 1) \varepsilon \end{aligned}$$

where $\|\Phi\|$ means the maximal value of $|\Phi|$. Now we can conclude that

$$\frac{\log T(n, \varepsilon)}{\log 2^n} \leq O\left(\frac{\log \varepsilon}{n}\right) + O(-\varepsilon \log \varepsilon) + \sup_{p \in A_k, |\varphi(p) - \alpha| < (2^k \|\Phi\| + 1)\varepsilon} H(p)$$

To finish the proof, let $n \rightarrow \infty$ then $\varepsilon \rightarrow 0$. ■

6. PROOF OF THEOREM 1

6.1. Lemmas

Let $n \geq k \geq 2$ be two integers, let $\omega = (x_j)_{j=1}^n \in \Sigma_n$. It is natural to introduce the following quantities. For $1 \leq i \leq k$, define

$$P_i(\omega) = \text{Card}\{1 \leq j \leq n - i + 1 : x_j \cdots x_{j+i-1} = 1\}$$

It is clear that $P_i(\omega)$ is the number of 1-blocks $1 \cdots 1$ of length i in the sequence ω . Here by 1-block (of length i) in $\omega = (x_i)_{i=1}^n$ we mean the words of the form $(x_m)_{m=j}^{j+i-1}$ with $x_\ell = 1$ for all $j \leq \ell \leq j+i-1$. Such a 1-block will be said to be maximal if $x_{j-1} = x_{j+i} = 0$ (with convention $x_0 = 0$ and $x_{n+1} = 0$). We shall be interested in those ω which are the heads of points in $E(\alpha_1, \dots, \alpha_k)$. Our aim is to study the limit of $n^{-1}P_i(\omega)$ as n tends to the infinity.

In order to practically estimate $P_i(\omega)$ ($\omega \in \Sigma_n$), we introduce another system of k quantities which are defined as follows:

$$N^*(\omega) := \text{the number of 1's in } \omega$$

$$N_*(\omega) := \text{the number of maximal 1-blocks in } \omega$$

$$N_i(\omega) := \text{the number of maximal 1-blocks of length } i \text{ in } \omega \ (1 \leq i \leq k-2)$$

It is obvious that

$$\sum_{i=1}^{k-2} N_i(\omega) \leq N_*(\omega) \leq N^*(\omega)$$

Suppose $\omega \neq 1 \cdots 1, 0 \cdots 0$ (the two constant sequences). Observe that ω must be in one of the following four forms

$$\begin{aligned} &1^{t_1} 0^{s_1} 1^{t_2} 0^{s_2} \dots 1^{t_r} 0^{s_r}, && 0^{s_1} 1^{t_1} 0^{s_2} 1^{t_2} \dots 0^{s_r} 1^{t_r} \\ &1^{t_1} 0^{s_1} 1^{t_2} 0^{s_2} \dots 1^{t_r}, && 0^{s_1} 1^{t_1} 0^{s_2} 1^{t_2} \dots 0^{s_r} 1^{t_r} 0^{s_{r+1}} \end{aligned}$$

where 1^t means a 1-block of length t (≥ 1) and 0^s means a 0-block of length s (≥ 1). When ω is represented as above, we have the following expression for the interested quantities

$$N^*(\omega) = t_1 + \cdots + t_r$$

$$N_*(\omega) = r$$

$$N_i(\omega) = \text{Card}\{1 \leq j \leq r : t_j = i\} \quad (1 \leq i \leq k-2)$$

Lemma 5. Suppose $x \in (x_j) \in E(\alpha_1, \dots, \alpha_k)$. Let $\omega = (x_j)_{j=1}^n \in \{0, 1\}^n$. We have

$$N^*(\omega) = \alpha_1 n + o(n)$$

$$N_*(\omega) = (\alpha_1 - \alpha_2) n + o(n)$$

$$N_i(\omega) = (\alpha_i - 2\alpha_{i+1} + \alpha_{i+2}) n + o(n) \quad (1 \leq i \leq k-2)$$

Proof. Clearly $N^*(\omega) = P_1(\omega) = \alpha_1 n + o(n)$. Define a mapping $S: \{0, 1\}^n \rightarrow \{0, 1\}^n$ by

$$\omega = (x_1, x_2, \dots, x_n) \mapsto (x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n, x_n)$$

Let us observe the change of 1-blocks from ω to $S\omega$. A 1-block of length r in ω is reduced to a 1-blocks of length $r-1$ in $S\omega$, with perhaps one exception (when $x_n = 1$). This observation implies the following four equalities

$$P_{i+1}(\omega) = P_i(S\omega) + O(1)$$

$$N^*(S\omega) = N^*(\omega) - N_*(\omega) + O(1)$$

$$N_*(S\omega) = N_*(\omega) - N_1(\omega) + O(1)$$

$$N_i(\omega) = N_1(S^{i-1}\omega) + O(1)$$

By using the first equality and the fact $P_1(\omega) = N^*(\omega)$, we have

$$P_i(\omega) = P_1(S^{i-1}\omega) + O(1) = N^*(S^{i-1}\omega) + O(1)$$

This together with the second equality yields

$$P_i(\omega) = P_{i-1}(\omega) - N_*(S^{i-2}\omega) + O(1)$$

Consider now

$$P_2(\omega) = P_1(S\omega) + O(1) = N^*(\omega) - N_*(\omega) + O(1)$$

Since $P_2(\omega) = \alpha_2 n + o(n)$, we get

$$N_*(\omega) = (\alpha_1 - \alpha_2) n + o(n)$$

Let now $1 \leq i \leq k - 2$. Consider

$$\begin{aligned} P_{i+2}(\omega) &= N^*(S^{i+1}\omega) + O(1) \\ &= N^*(S^i\omega) - N_*(S^i\omega) + O(1) \\ &= [N^*(S^{i-1}\omega) - N_*(S^{i-1}\omega)] - [N_*(S^{i-1}\omega) - N_1(S^{i-1}\omega)] + O(1) \\ &= P_i(\omega) - 2N_*(S^{i-1}\omega) + N_i(\omega) + O(1) \\ &= P_i(\omega) - 2[P_i(\omega) - P_{i+1}(\omega)] + N_i(\omega) + O(1) \end{aligned}$$

This together with $P_i(\omega) = \alpha_i n + o(n)$ gives $N_i(\omega) = (\alpha_i - 2\alpha_{i+1} + \alpha_{i+2}) n + o(n)$. ■

Let $n \geq k \geq 2$. For k non-negative integers n^*, n_*, n_i ($1 \leq i \leq k - 2$), define

$$\begin{aligned} \mathcal{A}(n^*, n_*, n_1, \dots, n_k) \\ = \{ \omega \in \Sigma_n : (N^*(\omega), N_*(\omega), N_1, \dots, N_{k-2}(\omega)) = (n^*, n_*, n_1, \dots, n_{k-2}) \} \end{aligned}$$

Lemma 6. With the above notation, we have

$$\begin{aligned} \text{Card } \mathcal{A}(n^*, n_*, n_1, \dots, n_k) \\ = [2C_{n-n^*-1}^{n^*-1} + C_{n-n^*-1}^{n^*} + C_{n-n^*-1}^{n^*-1}] C_{n_*}^{n_1} C_{n_*-n_1}^{n_2} \cdots C_{n_*-n_1-\dots-n_{k-3}}^{n_{k-2}} \\ \cdot C_{n^*-(k-2)n_*(k-3)n_1+(k-4)n_2+\dots+n_{k-3}}^{n_*-n_1-n_2-\dots-n_{k-2}} \end{aligned}$$

Proof. Let's first remark the following elementary fact. Let m, n be two positive integers. Then there are exactly C_{n-1}^{m-1} different positive integer solutions for the equation $x_1 + \dots + x_m = n$. Consequently, there are C_{n-km-1}^{m-1} different integer solutions for

$$y_1 + \dots + y_m = n, \quad y_i > k \quad \text{for } 1 \leq i \leq m$$

We account first all ω in the form $1^t 0^{s_1} 1^t 0^{s_2} \dots 1^t 0^{s_r}$. To get all such ω , we divide $n - n^*$ 0's into n_* groups of respective sizes s_1, s_2, \dots, s_{n_*} . Then we divide n^* 1's into n_* groups of respective sizes $t_1, t_2, \dots, t_{n_*}, n_i$ of which

are equal to i for each $1 \leq i \leq k-2$. So, the number of those ω is just the number of solutions of the following system

$$\begin{aligned} 1^\circ. & \quad s_1 + s_2 + \dots + s_{n_*} = n - n^*, \quad s_i \geq 1 \quad (1 \leq i \leq n_*); \\ 2^\circ. & \quad t_1 + t_2 + \dots + t_{n_*} = n^* \end{aligned}$$

under the condition that among these n_* integers t_1, \dots, t_{n_*} , there are n_1 integers taking value 1, n_2 integers taking value 2, ..., n_{k-2} integers taking value $k-2$; the remaining $n_* - (n_1 + \dots + n_{k-2})$ integers, if any, take values greater than $k-2$ and their sum is $n_* - (n_1 + 2n_2 + (k-2)n_{k-2})$.

According to the remark, there are $C_{n-n^*-1}^{n_*-1}$ solutions (s_1, \dots, s_{n_*}) satisfying 1° . The number of the solutions (t_1, \dots, t_{n_*}) of 2° is equal to

$$\begin{aligned} & C_{n_*}^{n_1} C_{n_*-n_1}^{n_2} \dots C_{n_*-n_1-\dots-n_{k-3}}^{n_{k-2}} \\ & \times C_{n^*-(n_1+2n_2+\dots+(k-2)n_{k-2})-(k-2)(n_*-n_1-\dots-n_{k-2})-1}^{n_*-n_1-\dots-n_{k-2}-1} \end{aligned}$$

The last factor in the above expression is obtained like this: After have arranged 1-blocks of lengths 1, 2, ..., $k-2$ there are yet $n^* - (n_1 + 2n_2 + \dots + (k-2)n_{k-2})$ 1's. They are then divided into $n_* - (n_1 + \dots + n_{k-2})$ groups each of which is of size strictly greater than $k-2$. By the remark, the number of all possibilities is the number of solutions of the equation

$$z_1 + \dots + z_{n_*-(n_1+\dots+n_{k-2})} = n^* - (n_1 + 2n_2 + \dots + (k-2)n_{k-2})$$

Thus the cardinal of the subset of elements of the form $1^{t_1}0^{s_1}1^{t_2}0^{s_2} \dots 1^{t_r}0^{s_r}$ in $\mathcal{A}(n^*, n_*, n_1, \dots, n_k)$ is equal to

$$\begin{aligned} & C_{n-n^*-1}^{n_*-1} C_{n_*}^{n_1} C_{n_*-n_1}^{n_2} \dots C_{n_*-n_1-\dots-n_{k-3}}^{n_{k-2}} \\ & \times C_{n^*-(k-2)n_*(k-3)n_1+(k-4)n_2+\dots+n_{k-3}}^{n_*-n_1-n_2-\dots-n_{k-2}} \end{aligned}$$

In a similar way, we can consider the other three forms

$$0^{s_1}1^{t_1}0^{s_2}1^{t_2} \dots 0^{s_r}1^{t_r}, \quad 1^{t_1}0^{s_1}1^{t_2}0^{s_2} \dots 1^{t_r}, \quad 0^{s_1}1^{t_1}0^{s_2}1^{t_2} \dots 0^{s_r}1^{t_r}0^{s_{r+1}}$$

and obtain the desired result. ■

Lemma 7. Let $n \geq m \geq \ell$ be three non-negative integers, then

$$\frac{1}{n} \log C_m^\ell = L\left(\frac{m}{n}\right) - L\left(\frac{\ell}{n}\right) - L\left(\frac{m-\ell}{n}\right) + O\left(\frac{\log n}{n}\right)$$

where $L(x) = x \log x$.

Proof. By the following form of Stirling formula

$$\log p! = p \log p - p + \frac{1}{2} \log p + O(1)$$

we have

$$\begin{aligned} \log C_m^\ell &= \log m! - \log \ell! - \log(m - \ell)! \\ &= m \log m - \ell \log \ell - (m - \ell) \log(m - \ell) \\ &\quad + \frac{1}{2} [\log m - \log \ell - \log(m - \ell)] + O(1) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{n} \log C_m^\ell &= \frac{1}{n} [m \log m - \ell \log \ell - (m - \ell) \log(m - \ell)] + O\left(\frac{\log n}{n}\right) \\ &= \frac{m}{n} \log \frac{m}{n} - \frac{\ell}{n} \log \frac{\ell}{n} - \frac{m - \ell}{n} \log \frac{m - \ell}{n} + O\left(\frac{\log n}{n}\right) \quad \blacksquare \end{aligned}$$

For $n \geq k \geq 2$ and $\varepsilon > 0$, let $F(n, \varepsilon)$ be the set of $\omega \in \Sigma_n$ such that

$$\begin{aligned} |N^*(\omega) - \alpha_1 n| &\leq \varepsilon n \\ |N_*(\omega) - (\alpha_1 - \alpha_2) n| &\leq \varepsilon n \\ |N_i(\omega) - (\alpha_i - 2\alpha_{i+1} + \alpha_{i+2}) n| &\leq \varepsilon n \quad (1 \leq i \leq k - 2) \end{aligned}$$

and let

$$f(n, \varepsilon) = \text{Card } F(n, \varepsilon)$$

Lemma 8. If $(\alpha_1, \alpha_2, \dots, \alpha_k)$ satisfies the convexity condition (1) in Theorem 1, then

$$\lim_{\varepsilon \downarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{\log f(n, \varepsilon)}{\log 2^n} = \lim_{\varepsilon \downarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{\log f(n, \varepsilon)}{\log 2^n} = A(\alpha_1, \alpha_2, \dots, \alpha_k)$$

Proof. Without loss of generality, we give only a proof for the case $k = 3$. For $n \geq 3$ and $\frac{1}{2} \geq \varepsilon > 0$, let n^*, n_*, n_1 be any integers satisfying

$$\begin{aligned} |n^* - \alpha_1 n| &\leq \varepsilon n \\ |n_* - (\alpha_1 - \alpha_2) n| &\leq \varepsilon n \\ |n_1 - (\alpha_1 - 2\alpha_2 + \alpha_3) n| &\leq \varepsilon n \end{aligned}$$

Let $V(n^*, n_*, n_1) = \text{Card } \mathcal{A}(n^*, n_*, n_1)$. By Lemma 6, we have

$$V(n^*, n_*, n_1) = [2C_{n-n^*-1}^{n^*-1} + C_{n-n^*-1}^{n^*} + C_{n-n^*-1}^{n^*-2}] \cdot C_{n^*}^{n_1} \cdot C_{n^*-n_*}^{n^*-n_1}$$

It is easy to see that $V(n^*, n_*, n_1) > 0$ only when $n \geq n^* \geq n_* \geq n_1$, $n \geq n^* + n_*$, $n^* - 2n_* + n_1 \geq 0$ and when n is great enough, there are always such n^*, n_*, n_1 . By Lemma 7,

$$\begin{aligned} \frac{1}{n} \log V(n^*, n_*, n_1) &= \frac{1}{n} \log(C_{n-n^*}^{n^*} \cdot C_{n^*}^{n_1} \cdot C_{n^*-n_*}^{n^*-n_1}) + O\left(\frac{\log n}{n}\right) \\ &= L\left(\frac{n^* - n_*}{n}\right) + L\left(\frac{n - n^*}{n}\right) - L\left(\frac{n_1}{n}\right) - L\left(\frac{n - n^* - n_*}{n}\right) \\ &\quad - 2L\left(\frac{n_* - n_1}{n}\right) - L\left(\frac{n^* - 2n_* + n_1}{n}\right) + O\left(\frac{\log n}{n}\right) \end{aligned}$$

Since

$$f(n, \varepsilon) = \sum_{n^*, n_*, n_1} V(n^*, n_*, n_1) \leq n^3 \sup_{n^*, n_*, n_1} V(n^*, n_*, n_1) \leq n^3 f(n, \varepsilon)$$

it follows that

$$\frac{\log f(n, \varepsilon)}{n} = \sup_{n^*, n_*, n_1} \frac{1}{n} \log V(n^*, n_*, n_1) + O\left(\frac{\log n}{n}\right)$$

Let $n \uparrow \infty$ and then $\varepsilon \downarrow 0$, since L is continuous, it follows that

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \overline{\lim}_{n \rightarrow \infty} \frac{\log f(n, \varepsilon)}{\log 2^n} &= \lim_{\varepsilon \downarrow 0} \underline{\lim}_{n \rightarrow \infty} \frac{\log f(n, \varepsilon)}{\log 2^n} \\ &= \frac{1}{\log 2} [L(\alpha_2) + L(1 - \alpha_1) - 2L(\alpha_2 - \alpha_3) - L(\alpha_3) \\ &\quad - L(1 - 2\alpha_1 + \alpha_2) - L(\alpha_1 - 2\alpha_2 + \alpha_3)] \\ &= A(\alpha_1, \alpha_2, \alpha_3) \quad \blacksquare \end{aligned}$$

6.2. Proof of Theorem 1

The basic idea is the same as that for proving Theorem 2. But here we are going to estimate directly the dimension without using Gibbs measures.

First we prove that $\dim_P E(\alpha_1, \dots, \alpha_k) \leq A(\alpha_1, \dots, \alpha_k)$. Let $\varepsilon > 0$ be an arbitrary positive number. If $x = (x_n) \in E(\alpha_1, \dots, \alpha_k)$, then $(x_1, \dots, x_n) \in F(n, \varepsilon)$ when n is large enough ($F(n, \varepsilon)$ was defined before Lemma 8). That means

$$E(\alpha_1, \dots, \alpha_k) \subset \bigcup_{\ell=1}^{\infty} G(\ell, \varepsilon)$$

where

$$G(\ell, \varepsilon) = \{y = (y_n) \in \Sigma : \forall m \geq \ell, (y_1, \dots, y_m) \in F(m, \varepsilon)\}$$

Note that if $m \geq \ell$, $G(\ell, \varepsilon)$ can be covered by $f(m, \varepsilon)$ m -cylinders (see the definition of $f(m, \varepsilon)$). It follows that the upper box dimension of $G(\ell, \varepsilon)$ does not exceed $\overline{\lim}_{m \rightarrow \infty} \log f(m, \varepsilon) / \log 2^m$. By the σ -stability of the packing dimension, we have

$$\dim_P E(\alpha_1, \dots, \alpha_k) \leq \overline{\lim}_{m \rightarrow \infty} \frac{\log f(m, \varepsilon)}{\log 2^m} \quad (\forall \varepsilon > 0)$$

Letting $\varepsilon \rightarrow 0$, we get the desired inequality.

Now we are going to show the inverse inequality. $\forall \delta > 0$, by Lemma 8, there are a sequence of integers $\ell_j \uparrow \infty$ and a sequence of real numbers $\varepsilon_j \downarrow 0$ such that

$$f(\ell_j, \varepsilon_j) > 2^{(A - \delta/2)\ell_j}$$

As in the proof of Theorem 4, define new sequences $\{\ell_j^*\}$ and $\{\varepsilon_j^*\}$. Then let

$$n_j = f(\ell_j^*, \varepsilon_j^*), \quad c_j = 2^{-\ell_j^*} \quad (j \geq 1)$$

$$\Theta = \prod_{j=1}^{\infty} Y_j, \quad Y_j = F(\ell_j^*, \varepsilon_j^*)$$

We claim that $\Theta \subset E(\alpha_1, \dots, \alpha_k)$ and $\dim_H \Theta \geq A - \delta$. Suppose $x = (x_i) \in \Theta$. For any $n (> m_1)$, there is a unique integer $J(n)$ such that

$$\sum_{i=1}^{J(n)} \ell_i^* \leq n \leq \sum_{i=1}^{J(n)+1} \ell_i^*$$

Let $\omega_n = x_1 \cdots x_n \in \Sigma_n$. Then

$$\begin{aligned} N^*(\omega_n) &\leq \ell_1^*[\alpha_1 + \varepsilon_1^*] + \cdots + \ell_{J(n)+1}^*[\alpha_1 + \varepsilon_{J(n)+1}^*] \\ N^*(\omega_n) &\geq \ell_1^*[\alpha_1 - \varepsilon_1^*] + \cdots + \ell_{J(n)}^*[\alpha_1 - \varepsilon_{J(n)}^*] \end{aligned}$$

It follows that

$$\lim_{n \rightarrow \infty} \frac{N^*(\omega_n)}{n} = \alpha_1$$

Observe that

$$N_*(\omega_n) \leq \ell_1^*[\alpha_1 - \alpha_2 + \varepsilon_1^*] + \cdots + \ell_{J(n)+1}^*[\alpha_1 - \alpha_2 + \varepsilon_{J(n)+1}^*]$$

$$N_*(\omega_n) \geq \ell_1^*[\alpha_1 - \alpha_2 - \varepsilon_1^*] + \cdots + \ell_{J(n)}^*[\alpha_1 - \alpha_2 - \varepsilon_{J(n)}^*]$$

so,

$$\lim_{n \rightarrow \infty} \frac{N_*(\omega_n)}{n} = \alpha_1 - \alpha_2$$

In the same way we can prove that

$$\lim_{n \rightarrow \infty} \frac{N_i(\omega_n)}{n} = \alpha_i - 2\alpha_{i+1} + \alpha_{i+2} \quad (1 \leq i \leq k-2)$$

Thus we have proved that $\Theta \subset E(\alpha_1, \dots, \alpha_k)$.

The set Θ is a homogeneous Moran set, i.e., $\Theta \in \mathcal{M}([0, 1], \{n_j\}, \{c_j\})$. The same calculation in the proof of Theorem 4 gives $\dim_H E(\alpha_1, \dots, \alpha_k) \geq A - \delta/2$, ($\forall \delta > 0$). ■

7. APPLICATIONS OF THEOREM 3

In this section, we first present an algorithm for computing the pressure of an energy function which depends only on a finite number of coordinates. Then we apply Theorem 3 to some special cases.

7.1. Calculation of the Pressure Function

For a function $g: \Sigma \rightarrow \mathbb{R}^+$, let

$$p(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \log 2^n \int_{\Sigma} \prod_{j=0}^{n-1} g(T^j x) dx$$

Then $P_{\Phi}(\beta) = p(\exp\langle \beta, \Phi \rangle)$. We are going to compute $p(g)$ for $g(x)$ depending only on finite number of coordinates of x .

Case I. $g(x) = g(x_1)$. Since

$$2^n \int_{\Sigma} \prod_{j=0}^{n-1} g(T^j x) dx = 2^n \sum_{x_1 \cdots x_n \in \{0, 1\}^n} g(x_1) g(x_2) \cdots g(x_n) \cdot \frac{1}{2^n} \\ = (g(0) + g(1))^n$$

it follows that $p(g) = \log(g(0) + g(1))$.

Case II. $g(x) = g(x_1, x_2)$. We have

$$2^n \int_{\Sigma} \prod_{j=0}^{n-1} g(T^j x) dx = \frac{1}{2} \sum_{x_1 \cdots x_{n+1} \in \{0, 1\}^{n+1}} g(x_1, x_2) g(x_2, x_3) \cdots g(x_n, x_{n+1})$$

Let now

$$G_n(0) = \sum_{x_1 \cdots x_n \in \{0, 1\}^n} g(x_1, x_2) g(x_2, x_3) \cdots g(x_{n-1}, x_n) g(x_n, 0)$$

$$G_n(1) = \sum_{x_1 \cdots x_n \in \{0, 1\}^n} g(x_1, x_2) g(x_2, x_3) \cdots g(x_{n-1}, x_n) g(x_n, 1)$$

Then, $G_n(0)$ and $G_n(1)$ satisfy the following recursive relation:

$$G_{n+1}(0) = G_n(0) g(0, 0) + G_n(1) g(1, 0)$$

$$G_{n+1}(1) = G_n(0) g(0, 1) + G_n(1) g(1, 1)$$

Therefore

$$\begin{pmatrix} G_n(0) \\ G_n(1) \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} g(0, 0) & g(1, 0) \\ g(0, 1) & g(1, 1) \end{pmatrix}$$

So,

$$2^n \int_{\Sigma} \prod_{j=0}^{n-1} g(T^j x) dx = \frac{1}{2} \|A^n\|_1$$

where $\|A\|_1$ denotes the norm of A defined by the sum of all absolute values of its entries. Thus we get $p(g) = \log \text{Spec}(A)$ where $\text{Spec}(A)$ denotes the spectral radius of A .

Case III. $g(x) = g(x_1, x_2, x_3)$. We have

$$2^n \int_{\Sigma} \prod_{j=0}^{n-1} g(T^j x) dx = \frac{1}{2^2} \sum_{x_1 \cdots x_{n+2} \in \{0, 1\}^{n+2}} g(x_1, x_2, x_3) \cdots g(x_n, x_{n+1}, x_{n+2})$$

For any $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, set

$$G_n(\varepsilon_1, \varepsilon_2) = \sum_{\substack{x_1 \cdots x_{n+2} \in \{0, 1\}^{n+2} \\ x_{n+1} = \varepsilon_1, x_{n+2} = \varepsilon_2}} g(x_1, x_2, x_3) \cdots g(x_n, x_{n+1}, x_{n+2})$$

Then $G_n(\varepsilon_1, \varepsilon_2)$ satisfy the following recursive relation:

$$\begin{aligned} G_{n+1}(0, 0) &= G_n(0, 0) g(0, 0, 0) + G_n(1, 0) g(1, 0, 0) \\ G_{n+1}(0, 1) &= G_n(0, 0) g(0, 0, 1) + G_n(1, 0) g(0, 1, 0) \\ G_{n+1}(1, 0) &= G_n(0, 1) g(0, 1, 0) + G_n(1, 1) g(1, 1, 0) \\ G_{n+1}(1, 1) &= G_n(0, 1) g(0, 1, 1) + G_n(1, 1) g(1, 1, 1) \end{aligned}$$

Thus

$$2^n \int_{\Sigma} \prod_{j=0}^{n-1} g(T^j x) dx = \frac{1}{2^2} \|A^n\|_1$$

and

$$p(g) = \log \text{Spec}(A)$$

where

$$A = \begin{pmatrix} g(0, 0, 0) & 0 & g(1, 0, 0) & 0 \\ g(0, 0, 1) & 0 & g(1, 0, 1) & 0 \\ 0 & g(0, 1, 0) & 0 & g(1, 1, 0) \\ 0 & g(0, 1, 1) & 0 & g(1, 1, 1) \end{pmatrix}$$

Case IV. $g(x) = g(x_1, \dots, x_k)$. In this general case, we can get

$$p(g) = \log \text{Spec}(A)$$

where $A = (a_{i, j})_{2^{k-1} \times 2^{k-1}}$ is defined as follows. Write

$$i = 1 + \sum_{\ell=1}^{k-1} \varepsilon_{\ell} 2^{k-1-\ell}, \quad j = 1 + \sum_{\ell=1}^{k-1} v_{\ell} 2^{k-1-\ell}, \quad (\varepsilon_{\ell}, v_{\ell} = 0 \text{ or } 1)$$

Define

$$a_{i, j} = \begin{cases} g(0, \varepsilon_1, \dots, \varepsilon_{k-1}), & \text{if } v_1 = 0, v_j = \varepsilon_{j-1} \text{ for } 2 \leq j \leq k-1 \\ g(1, \varepsilon_1, \dots, \varepsilon_{k-1}), & \text{if } v_1 = 1, v_j = \varepsilon_{j-1} \text{ for } 2 \leq j \leq k-1 \\ 0, & \text{otherwise} \end{cases}$$

7.2. Computation of $\dim E_{\Phi}(\alpha)$ by Theorem 3

In this section, we give two examples to show how the method provided by Theorem 3 allows us to compute the dimension of $E_{\Phi}(\alpha)$.

Example 1. $\Phi(x) = (x_1, x_1 x_2)$. By the algorithm we have just discussed above, we get

$$P_{\Phi}(\beta_1, \beta_2) = \log[1 + \exp(\beta_1 + \beta_2) + \sqrt{(1 - \exp(\beta_1 + \beta_2))^2 + 4 \exp(\beta_1)}] - \log 2$$

Write down

$$\frac{\partial P_{\Phi}}{\partial \beta_1} = \frac{1}{1 + e^{\beta_1 + \beta_2} + \sqrt{(1 - e^{\beta_1 + \beta_2})^2 + 4e^{\beta_1}}} \left(e^{\beta_1 + \beta_2} + \frac{e^{2\beta_1 + 2\beta_2} - e^{\beta_1 + \beta_2} + 2e^{\beta_1}}{\sqrt{(1 - e^{\beta_1 + \beta_2})^2 + 4e^{\beta_1}}} \right)$$

$$\frac{\partial P_{\Phi}}{\partial \beta_2} = \frac{1}{1 + e^{\beta_1 + \beta_2} + \sqrt{(1 - e^{\beta_1 + \beta_2})^2 + 4e^{\beta_1}}} \left(e^{\beta_1 + \beta_2} + \frac{e^{2\beta_1 + 2\beta_2} - e^{\beta_1 + \beta_2}}{\sqrt{(1 - e^{\beta_1 + \beta_2})^2 + 4e^{\beta_1}}} \right)$$

Now taking $\beta = (0, 0)$ gives us

$$\nabla P_{\Phi}(0, 0) = \left(\frac{1}{2}, \frac{1}{4} \right), \quad P_{\Phi}(0, 0) = \log 2$$

$$\dim_H E \left(\frac{1}{2}, \frac{1}{4} \right) = -\frac{1}{\log 2} [\langle \beta, \nabla P_{\Phi}(\beta) \rangle - P_{\Phi}(\beta)] = 1$$

It is a trivial result. Taking $\beta = (1, -1)$ gives us

$$\nabla P_{\Phi}(\beta) = \left(\frac{1}{2}, \frac{1}{2(1 + \sqrt{e})} \right), \quad P_{\Phi}(\beta) = \log(1 + \sqrt{e})$$

$$\dim_H E \left(\frac{1}{2}, \frac{1}{2(1 + \sqrt{e})} \right) = \frac{1}{\log 2} \left[\log(1 + \sqrt{e}) - \frac{1}{2} \left(1 - \frac{1}{1 + \sqrt{e}} \right) \right]$$

These results can be verified by the formula provided by Theorem 1.

Example 2. $\Phi(x) = (x_1, x_2(1 - x_1))$. Note that $\alpha = (\alpha_1, \alpha_2) \in L_{\Phi}$ means α_1 (resp. α_2) is the proportion of word “1” (resp. “01”) in dyadic development of points in $E_{\Phi}(\alpha)$. In this case, we have

$$P_{\Phi}(\beta_1, \beta_2) = \log[1 + \exp(\beta_1) + \sqrt{(1 - \exp(\beta_1))^2 + 4 \exp(\beta_1 + \beta_2)}] - \log 2$$

$$\frac{\partial P_{\Phi}}{\partial \beta_1} = \frac{1}{1 + e^{\beta_1} + \sqrt{(1 - e^{\beta_1})^2 + 4e^{\beta_1 + \beta_2}}} \left(e^{\beta_1} + \frac{e^{2\beta_1} - e^{\beta_1} + 2e^{\beta_1 + \beta_2}}{\sqrt{(1 - e^{\beta_1})^2 + 4e^{\beta_1 + \beta_2}}} \right)$$

$$\frac{\partial P_{\Phi}}{\partial \beta_2} = \frac{1}{1 + e^{\beta_1} + \sqrt{(1 - e^{\beta_1})^2 + 4e^{\beta_1 + \beta_2}}} \left(\frac{2e^{\beta_1 + \beta_2}}{\sqrt{(1 - e^{\beta_1})^2 + 4e^{\beta_1 + \beta_2}}} \right)$$

Taking $\beta = (0, 0)$ gives

$$\nabla P_{\Phi}(0, 0) = \left(\frac{1}{2}, \frac{1}{4}\right), \quad P_{\Phi}(0, 0) = \log 2$$

$$\dim_H E\left(\frac{1}{2}, \frac{1}{4}\right) = 1$$

Taking $\beta = (0, 1)$ gives

$$\nabla P_{\Phi}(\beta) = \left(\frac{1}{2}, \frac{\sqrt{e}}{2(1 + \sqrt{e})}\right), \quad P_{\Phi}(\beta) = \log(1 + \sqrt{e})$$

$$\dim_H E\left(\frac{1}{2}, \frac{\sqrt{e}}{2(1 + \sqrt{e})}\right) = \frac{1}{\log 2} \left[\log(1 + \sqrt{e}) - \frac{\sqrt{e}}{2(1 + \sqrt{e})} \right]$$

These results are not directly covered by Theorem 1, but by Theorem 2.

8. APPLICATIONS OF THEOREMS 1 AND 2

8.1. $\Phi(x) = \Phi(x_1)$

We have $L_{\Phi} = \overline{\Phi(0)\Phi(1)}$ (the segment joining $\Phi(0)$ and $\Phi(1)$). For $\alpha \in L_{\Phi}$, we have

$$\dim_H E_{\Phi}(\alpha) = \dim_P E_{\Phi}(\alpha) = -\gamma \log_2 \gamma - (1 - \gamma) \log_2(1 - \gamma)$$

where

$$\gamma = \frac{|\alpha - \Phi(1)|}{|\Phi(1) - \Phi(0)|}, \quad 1 - \gamma = \frac{|\alpha - \Phi(0)|}{|\Phi(1) - \Phi(0)|}$$

The limit set is clear because $\mathcal{A}_1 = \{p = (p_0, p_1) : p_0, p_1 \geq 0, p_0 + p_1 = 1\}$. So, $\alpha = (\alpha_1, \alpha_2) = \varphi(p)$ iff $\alpha_1 = \gamma$ and $\alpha_2 = 1 - \gamma$. Now it suffices to write down the formula according to Theorem 2.

Let $p = (a, b, b, c) \in \mathcal{A}_2$. It suffices to observe that $\alpha = \varphi(p)$ iff $\alpha_1 = a$, $\alpha_2 = \alpha_3 = b$ and $\alpha_4 = c$. This formula was obtained by Billingsley in ref. 2. But, in ref. 2, it was not mentioned what happens when $\alpha \in \mathcal{A}_2 \setminus \mathcal{A}_2^+$.

Example 4. If $\Phi(x) = (x_1, x_1 x_2)$, then L_Φ is the triangle in \mathbb{R}^2 of vertexes $(0, 0)$, $(1/2, 0)$ and $(1, 1)$ and for $\alpha = (\alpha_1, \alpha_2) \in L_\Phi$ we have

$$\begin{aligned} \dim_H E_\Phi(\alpha) &= \dim_P E_\Phi(\alpha) \\ &= \alpha_1 \log_2 \alpha_1 + (1 - \alpha_1) \log_2(1 - \alpha_1) \\ &\quad - \alpha_2 \log_2 \alpha_2 - 2(\alpha_1 - \alpha_2) \log_2(\alpha_1 - \alpha_2) \\ &\quad - (1 - 2\alpha_1 + \alpha_2) \log_2(1 - 2\alpha_1 + \alpha_2) \end{aligned}$$

The set L_Φ is clear. Note that $\alpha = \varphi(p)$ means $\alpha_1 = b + c$ and $c = \alpha_2$. Recall that $a + 2b + c = 1$. From these three equations we get $a = 1 - 2\alpha_1 + \alpha_2$, $b = \alpha_1 - \alpha_2$, $c = \alpha_2$. Now it suffices to write down the formula in Theorem 2. This confirms Theorem 1 for $k = 2$.

Example 5. If $\Phi(x) = (x_1, (1 - x_1) x_2)$, then L_Φ is the triangle in \mathbb{R}^2 of vertexes $(0, 0)$, $(1/2, 1/2)$ and $(1, 0)$ and for $\alpha = (\alpha_1, \alpha_2) \in L_\Phi$ we have

$$\begin{aligned} \dim_H E_\Phi(\alpha) &= \dim_P E_\Phi(\alpha) \\ &= \alpha_1 \log_2 \alpha_1 + (1 - \alpha_1) \log_2(1 - \alpha_1) \\ &\quad - 2\alpha_2 \log_2 \alpha_2 - (\alpha_1 - \alpha_2) \log_2(\alpha_1 - \alpha_2) \\ &\quad - (1 - \alpha_1 - \alpha_2) \log_2(1 - \alpha_1 - \alpha_2) \end{aligned}$$

Example 6. If $\Phi(x) = x_1 x_2$, we have $L_\Phi = [0, 1]$ and for $\alpha \in [0, 1]$

$$\dim_H E_\Phi(\alpha) = \dim_P E_\Phi(\alpha) = H(p_\alpha)$$

where $p_\alpha = (2x_\alpha - 1 + \alpha, 1 - \alpha - x_\alpha, 1 - \alpha - x_\alpha, \alpha) \in \mathcal{A}_2$ where $x_\alpha \in [0, 1]$ is the solution of

$$x(1 - \alpha - x)^2 = (1 - x)(2x - 1 + \alpha)^2$$

Note that $\Phi(0, 0) = \Phi(0, 1) = \Phi(1, 0) = 0$ and $\Phi(1, 1) = 1$. Let $p = (a, b, b, c) \in \mathcal{A}_2$. Then $\alpha = \varphi(p)$ iff $\alpha = c$. It follows, from $a + 2b + c = 1$, that $L_\Phi = [0, 1]$. Note that

$$\begin{aligned}
 H(p) &= \log_2 \left(\frac{a+b}{a} \right)^a \left(\frac{a+b}{b} \right)^b \left(\frac{b+c}{b} \right)^b \left(\frac{b+c}{c} \right)^c \\
 &= \log_2 \frac{(a+b)^{a+b} (b+c)^{b+c}}{a^a b^{2b} c^c}
 \end{aligned}$$

Introduce the variable $x = a + b$. Since $a + 2b + c = 1$ and $c = \alpha$, we have $b + c = 1 - x$, $b = 1 - \alpha - x$ and $a = 2x - 1 + \alpha$. Consider H as a function of x , which is concave. x_α must be the solution of $dH/dx = 0$. However

$$\frac{dH}{dx} = \log_2 \frac{x(1 - \alpha - x)^2}{(1 - x)(2x - 1 + \alpha)^2}$$

A. Bisbas⁽³⁾ studied the above situation and obtained a different formula. More general situation in the next subsection was also studied by A. Bisbas.⁽⁴⁾ We shall obtain a result for it as a consequence of Theorem 1.

8.3. $\Phi_k(x) = x_1 x_2 \cdots x_k$

Our aim is to study the set $E_{\Phi_k}(\beta)$ where $\Phi_k(x) = x_1 x_2 \cdots x_k$ is the product (not the concatenation) of the first k coordinates. $E_{\Phi_k}(\beta)$ is thus the set of points x which have βn 1-blocks of length k in their first n coordinates.

Introduce the vector valued function $\Phi: \Sigma \rightarrow \mathbb{R}^k$ defined by $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_k)$ where $\Phi_j(x) = x_1 x_2 \cdots x_j$ ($1 \leq j \leq k$). By Theorem 1, we know that

$$L_\Phi = \{(\alpha_1, \alpha_2, \dots, \alpha_k) : 1 = \alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_k \geq 0, \{\alpha_j\}_{j=0}^k \text{ is convex}\}$$

For $0 \leq \beta \leq 1$, denote by $D_{\beta,k}$ the section of L_Φ defined by

$$D_{\beta,k} = \{(\alpha_1, \alpha_2, \dots, \alpha_{k-1}) : 1 = \alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_k = \beta, \{\alpha_j\}_{j=0}^k \text{ is convex}\}$$

It is clear that $D_{1,k}$ is a singleton and $\dim E_{\Phi_k}(1) = \dim E_\Phi(1, 1, \dots, 1) = 0$. If $0 \leq \beta < 1$, $D_{\beta,k}$ is a convex set of dimension $k - 1$.

Theorem 5. For $0 \leq \beta \leq 1$, we have

$$\dim_H E_{\Phi_k}(\beta) = \dim_P E_{\Phi_k}(\beta) = \sup_{D_{\beta,k}} A(\alpha_1, \dots, \alpha_{k-1}, \beta)$$

where A is the function defined in Theorem 1. If $0 \leq \beta < 1$, the supremum is attained at the unique solution $(\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1})$ of the system of equations

$$\frac{\partial}{\partial \alpha_j} A(\alpha_1, \dots, \alpha_{k-1}, \beta) = 0 \quad (1 \leq j \leq k-1)$$

Proof. Since $E_{\Phi}(\alpha_1, \dots, \alpha_{k-1}, \beta) \subset E_{\Phi_k}(\beta)$, the supremum is bounded by $\dim_H E_{\Phi_k}(\beta)$. According to Theorem 2, there is a vector probability $p \in \Delta_k$ such that

$$\sum_{x_1, \dots, x_k} p(x_1, \dots, x_k) \Phi_k(x_1, \dots, x_k) = \beta$$

$$\dim_H E_{\Phi_k}(\beta) = H(p)$$

Denote

$$\hat{\alpha}_j = \sum_{x_1, \dots, x_k} p(x_1, \dots, x_k) \Phi_j(x_1, \dots, x_k) \quad (1 \leq j \leq k-1)$$

Again by Theorem 2, we have

$$\dim_H E_{\Phi}(\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1}, \beta) \geq H(p)$$

It is sure that $(\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1}) \in D_{\beta, k}$ because $(\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1}, \beta) \in L_{\Phi}$. Combining the last two expressions involving $H(p)$ gives the desired formula. We have seen that $(\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1})$ is the maximal point of $A(\cdot, \beta)$. Since $A(\cdot, \beta)$ is a differentiable strictly concave function in $D_{\beta, k}$, if we can prove that $A(\cdot, \beta)$ doesn't attain its maximum on the boundary of $D_{\beta, k}$, $(\hat{\alpha}_1, \dots, \hat{\alpha}_{k-1})$ must be the unique solution of the system. According to the definition of A , it can be checked that for any boundary point $\bar{\alpha} \in D_{\beta, k}$, the directional derivative

$$\frac{dA(\bar{\alpha}, \beta)}{d\ell} = +\infty$$

where ℓ is a direction pointing to the interior of $D_{\beta, k}$. This implies that $A(\cdot, \beta)$ doesn't attain its maximum on the boundary of $D_{\beta, k}$. ■

8.4. $\Phi(x) = (\Phi_{n_1}(x_1, x_2, \dots, x_k), \dots, \Phi_{n_s}(x_1, x_2, \dots, x_k))$

In the same way as above, we can deal with

$$\Phi(x) = (\Phi_{n_1}(x_1, x_2, \dots, x_k), \dots, \Phi_{n_s}(x_1, x_2, \dots, x_k))$$

where $1 \leq n_1 < \dots < n_s \leq k$ and $\Phi_j(x) = x_1 x_2 \dots x_j$ ($1 \leq j \leq k$). For example, when $\Phi(x) = (x_1 x_2, x_1 x_2 x_3 x_4)$, for any $(\alpha, \beta) \in L_\Phi$ we have $\dim E_{\alpha, \beta} = A(\hat{x}, \alpha, \hat{y}, \beta)$ where (\hat{x}, \hat{y}) is the solution of the system

$$\frac{\partial}{\partial x} A(x, \alpha, y, \beta) = 0, \quad \frac{\partial}{\partial y} A(x, \alpha, y, \beta) = 0.$$

More generally, Theorem 2 may be used to deal with any function Φ which depends only upon a finite number of coordinates. Consider just an example.

Example 7. $\Phi(x) = x_2 - 2x_1 x_3 + 3x_1 x_2 x_3$.

Note that A_3 is a 4-dimensional convex. For $p \in A_3$, let

$$\begin{aligned} x &= p(0, 0, 0) + p(0, 0, 1), & y &= p(0, 1, 0) + p(0, 1, 1) \\ z &= p(0, 0, 1), & w &= p(0, 1, 1) \end{aligned}$$

It may be checked that

$$\varphi(p) = 2 - 2x + 2y + 2z - w$$

Suppose $\alpha \in L_\Phi$ with $\alpha = \varphi(p)$. We get $w = 2 - 2x + 2y + 2z - \alpha$. Thus $H(p) = f(x, y, z)$ where

$$f(x, y, z) = F(x - z, z) + F(y - w, w) + F(z, y - z) + F(w, 1 - x - 2y - w)$$

with $F(a, b) = h(a) + h(b) - h(a + b)$. It can be proved that for $\alpha \in L_\Phi$, we have

$$\dim E_\Phi(\alpha) = \sup_{p \in A_3, \varphi(p) = \alpha} H(p) = f(\hat{x}, \hat{y}, \hat{z})$$

where $(\hat{x}, \hat{y}, \hat{z})$ is the unique solution of the system

$$\frac{\partial}{\partial x} f(x, y, z) = 0, \quad \frac{\partial}{\partial y} f(x, y, z) = 0, \quad \frac{\partial}{\partial z} f(x, y, z) = 0$$

We point out that $F(a, b)$ admits $+\infty$ as its directional derivative at a boundary point with a direction pointing to the interior of its domain of definition.

9. GENERALIZATIONS

Theorems 2, 3 and 4 hold for a symbolic space with more than two symbols, say m symbols. It suffices to replace \log_2 by \log_m in the statements of the theorems. More generally, these theorems can be generalized to transitive subshifts of finite type. Let $\Sigma = \{1, 2, \dots, m\}^{\mathbb{N}}$ ($m \geq 2$) and T be the shift on Σ . Let $A = (a_{i,j})$ be a $m \times m$ matrix with $a_{i,j} \in \{0, 1\}$. Define

$$\Sigma_A = \{(x_n) \in \Sigma : a_{x_n, x_{n+1}} = 1 \ \forall n \geq 1\}$$

Note that $T\Sigma_A \subset \Sigma_A$. The system (Σ_A, T) is called a *subshift of finite type*. Suppose further that all the entries of A^M are strictly positive for some $M \geq 1$. Then the subshift is said to be (*topologically*) *transitive*.

The statement of Theorem 4 doesn't change for transitive subshifts. But in the definition of $f(\alpha, n, \varepsilon)$, by a n -cylinder $I(x_1, \dots, x_n)$ we means

$$I(x_1, \dots, x_n) = \{(y_n) \in \Sigma_A : y_1 = x_1, \dots, y_n = x_n\}$$

So, $I(x_1, \dots, x_n)$ is empty if $a_{x_j, x_{j+1}} = 0$ for some $1 \leq j < n$.

Let $\Sigma_{A,k}$ be the set of all sequences (x_1, \dots, x_n) such that $a_{x_j, x_{j+1}} = 1$ for all $1 \leq j < n$. Let $\Delta_k = \Delta_{A,k}$ (associated to Σ_A) be the set of probability vectors p defined on $\Sigma_{A,k}$ such that

$$\sum_i p(x_1, \dots, x_{k-1}, i) = \sum_j p(j, x_1, \dots, x_{k-1})$$

where the first sum is taken over i 's such that $a_{x_{k-1}, i} = 1$ and the second sum is taken over j 's such that $a_{j, x_1} = 1$. For a function $\Phi: \Sigma_A \rightarrow \mathbb{R}^d$ depending only upon the first k coordinates, define $\varphi: \Delta_k \rightarrow \mathbb{R}^d$

$$\varphi(p) = \sum_{p \in \Sigma_{A,k}} p(x) \Phi(x)$$

Formally, Theorem 2 also holds in the case of transitive subshift. But the function H is

$$H(p) = \sum_{x_1, \dots, x_k} p(x_1, \dots, x_k) \log_m \frac{\sum_i p(x_1, \dots, x_{k-1}, i)}{p(x_1, \dots, x_k)}$$

It is understood that $p(x) = 0$ if $x \notin \Sigma_{A,k}$.

Since $\Sigma_{A,k}$ is transitive, for any $\omega = (x_j)_{j=1}^n \in \Sigma_{A,n}$ and any $1 \leq x_0 \leq m$, there are $1 \leq y_1, \dots, y_{M-1} \leq m$ such that

$$(x_1, \dots, x_n, y_1, \dots, y_{M-1}, x_0) \in \Sigma_{A, n+M}$$

We call $\bar{\omega} = (x_1, \dots, x_n, y_1, \dots, y_{M-1})$ an *extension* of ω joining x_0 . The essential change in the proofs of Theorems 2 and 4 is to replace finite sequences which appear in different constructions of infinite sequences by their extensions. For example, in Step 2 of the proof of Theorem 4, we define

$$\omega = \underbrace{\bar{\omega}_1 \cdots \bar{\omega}_1 \bar{\omega}'_1}_{m_1} \underbrace{\bar{\omega}_2 \cdots \bar{\omega}_2 \bar{\omega}'_2}_{m_2} \underbrace{\bar{\omega}_3 \cdots \bar{\omega}_3 \bar{\omega}'_3}_{m_3} \cdots$$

where $\bar{\omega}_1$ is an extension of $x^{(1)}|_{n_1}$ joining $x^{(1)}|_1$ but $\bar{\omega}'_1$ is an extension of $x^{(1)}|_{n_1}$ joining $x^{(2)}|_1$; $\bar{\omega}_2$ is an extension of $x^{(2)}|_{n_2}$ joining $x^{(2)}|_1$ but $\bar{\omega}'_2$ is an extension of $x^{(2)}|_{n_2}$ joining $x^3|_1$; and so on.

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